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ASTROPHYSICS RESEARCH CORPORATION
10889 WILSHIRE BLVD LOS ANGELES, CALIFORNIA 90024

RADAR CROSS SECTIONS OF INHCMOGENEOUS PLASMA SPHERES PART 1

BY
VICTOR A. ERMA

2 APRIL 1965

PREPARED FOR
OFFICE OF NAVAL RESEARCH
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ASTROPHYSICS RESEARCH CORPORATION
10889 Wilshire Boulevard
Los Angeles, California 90024

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ABSTRACT

The question whether the measurement of radar cross-sections at different frequencies provides a useful diagnostic tool for ascertaining the electron density distribution of spherically symmetric plasma clouds is investigated. This is accomplished by comparing the calculated radar cross-sections of characteristic plasmas with increasing and decreasing refractive index. Exact analytical expressions for the radar cross-sections of several typical plasma spheres with increasing and decreasing refractive index are calculated. The calculations are based on an exact wave treatment of the scattering problem. Part I of the present report contains the exact analytical results obtained, while Part II will be devoted to the numerical evaluation of these results, as well as to asymptotic expressions for the limiting cases of high and low frequencies.

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I. INTRODUCTION

In many areas of practical importance, it is of great interest to be able to ascertain the electron density distribution of inhomogeneous plasmas which are not accessible to direct measurement. For example, such plasmas or clouds of ionization might arise from artificial perturbations in the ionosphere or from nuclear detonations in the atmosphere. An accurate knowledge of the electron density distribution in such plasmas would be of great aid in understanding the phenomena, e.g. the nature of the reactions and their respective reaction rates, taking place inside the plasmas.

The ultimate purpose of our present investigation is to determine to what extent information concerning the electron density distribution of plasmas can be obtained by means of ground-based radar cross-section measurements carried out at different frequencies. To begin with, we shall be concerned only with inhomogeneous plasma spheres with a spherically symmetric electron density distribution. Accordingly, we

necessarily limit our considerations to non-turbulent plasmas. Moreover, we shall assume that all electromagnetic quantities (e.g. dielectric constant, conductivity, refractive index) describing the plasma are likewise spherically symmetric scalars. We are thus restricting ourselves to the case where the background magnetic field of the Earth with its attendant anisotropy is negligible. From a practical point of view, our results will then be valid for plasmas for which the electron collision frequency greatly exceeds the Larmor frequency within the plasma.

Thus, the general problem to which we address ourselves is the determination of the radar cross-sections of inhomogeneous (albeit spherically symmetric) plasmas embedded in a medium of uniform electromagnetic properties. This outside medium need not necessarily possess the electromagnetic properties of the vacuum; our treatment is equally applicable to a plasma cloud situated in the ionosphere, as long as the electromagnetic properties of the surrounding ionosphere can be considered as approximately uniform.

Our approach to the problem will consist of an <u>exact wave treatment</u> of the scattering of a plane wave from the plasma under consideration, based on Maxwell's equations without the introduction of any approximations.

While the overall motivation of the present research is thus to ascertain whether the measurement of radar cross-sections can be considered a useful diagnostic tool for determining the electron density profile of spherically symmetric plasmas, the present study does not attempt to answer this question in its most general form. We shall be concerned here only with a preliminary investigation designed to yield a comparison of the radar cross-sections of spherical plasmas with increasing and decreasing complex refractive index, as a function of radial distance from the origin. The results of this investigation will show whether the two characteristic cases of increasing and decreasing refractive index give rise to distinctly different radar cross-sections as a function of frequency, such that we may conclude that the measurement of radar cross-sections holds promise as a diagnostic tool and therefore warrants further investigation. Future planned work includes the possibility of increasing the information obtained about the electron density profile of plasmas by means of measurements carried out by ground-based receivers at different locations. This would involve a calculation of the full differential scattering cross-sections of the plasmas. In the present preliminary work, however, we shall be concerned only with proper radar cross-sections.

Toward this end, we shall here obtain exact analytical expressions for the radar cross-sections of the four special cases illustrated in Fig. 1 below.

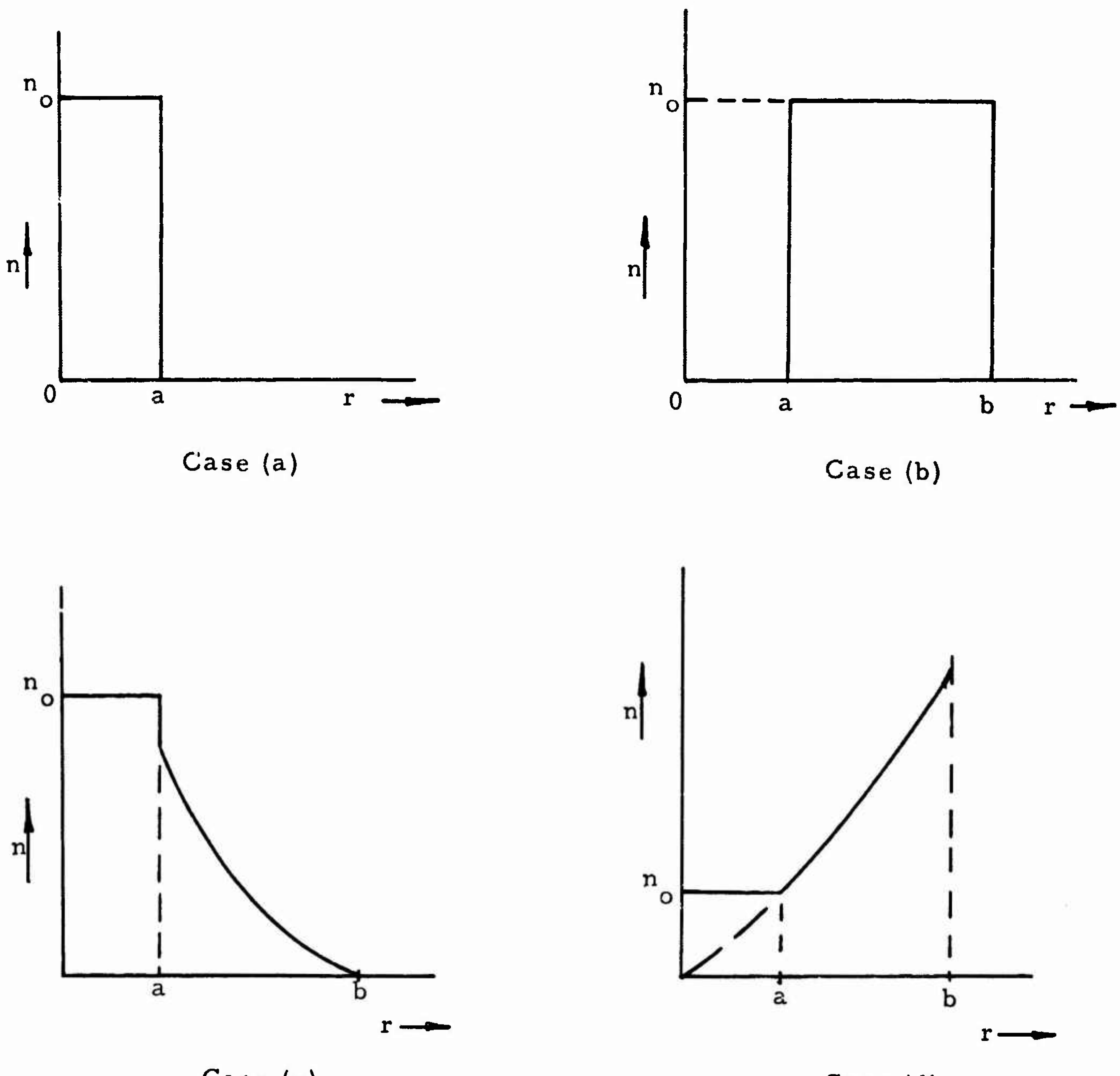

FIGURE 1

In Figure 1, we have plotted schematically the refractive index n as a function of radial distance r for the four general cases we shall consider. A few explanatory remarks may be called for. To begin with, the plots in Fig. 1 are only schematic, since n is in general a complex quantity (we include absorption in our analysis). In all cases, the space outside the sphere $(r > b)$ is considered to have uniform electromagnetic properties (n = constant). Cases (a) and (b) represent the extreme cases where the electrons are concentrated with constant density in the center and in the outside of the spherical plasma, respectively. Cases (c) and (d) represent a more continuous variation. In both cases (c) and (d), the presence of the discontinuity in n at $r=a$ represents the most general case considered. The analysis carried out below includes the special cases of no discontinuity; thus, we have also included the cases where in case (c), $n(a) = n_o$, and in case (d), n decreases continuously to the constant value of the outside medium at $r=0$, as shown by the dotted curve. Finally, while we have shown the refractive index as greater than unity in the schematic diagrams of Fig. 1, our analysis applies equally well to refractive indices which are less than unity or negative.

The subject matter of our investigation conveniently separates into two portions. Part I, constituting the present report, is concerned with

the derivation of analytical expressions for the radar cross-sections of the various characteristic plasma spheres of interest. While much of the work reported here leads to results in agreement with other authors (Wyatt,[1] Levine and Kerker[2]), the work of these authors contains a considerable number of errors and serious ambiguities, such that it was felt advisable to rederive the entire formalism in detail. An adequate outline of the work and results contained in the present report is provided by the Table of Contents. Part II will be concerned with the numerical reduction as well as with asymptotic approximations of the analytical results obtained in Part I.

II. MAXWELL'S EQUATIONS FOR AN INHOMOGENEOUS MEDIUM

We shall use MKS units throughout the present work. The two basic Maxwell equations may then be written in the form

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \tag{1}$$

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \quad , \tag{2}$$

where the various electromagnetic vectors are related by the so-called constitutive relations

$$\vec{B} = \mu \vec{H} \tag{3}$$

$$\vec{D} = \epsilon \vec{E} \tag{4}$$

and by Ohm's law:

$$\vec{j} = \sigma \vec{E} \tag{5}$$

We consider the case where the magnetic permittivity of the medium is that of the vacuum ($\mu = \mu_o$), while the dielectric constant ϵ and the conductivity σ may be functions of position within the medium. We further assume that the time variation of all electromagnetic quantities has the form $e^{-i\omega t}$. Substituting this time variation, as well as Eq. (5), into Eqs. (1) and (2), the two basic Maxwell equations may be written in the form

$$\nabla \times \vec{E} = k_2 \vec{H} \tag{6}$$

$$\nabla \times \vec{H} = -k_1 \vec{E} \tag{7}$$

where k_2 and k_1 are given by

$$k_2 = i \mu_o \omega \tag{8}$$

$$k_1 = i\omega \left(\epsilon + \frac{i\sigma}{\omega} \right) \tag{9}$$

We further define the propagation constant k by means of

$$k^2 = -k_1 k_2 = \mu_o \omega^2 \left(\epsilon + \frac{i\sigma}{\omega} \right) \tag{10}$$

It is important to note that with the assumptions made in our case, k_2 is a constant, while k_1 and k are in general functions of position.

It is customary to complete the set of Maxwell equations with two further equations which give expressions for the divergences of $\vec{B}$ and $\vec{E}$. The first of these is

$$\nabla \cdot \vec{B} = 0$$

(or equivalently)

$$\nabla \cdot \vec{H} = 0 \tag{11}$$

However, we observe from Eqs. (6) and (8) that Eq. (11) is not an independent equation, but follows automatically from Eq. (6), since the divergence of any curl vanishes identically. The case of the divergence of $\vec{D}$ is more subtle and has led to considerable confusion in the literature. In all cases, we may write

$$\nabla \cdot \vec{D} = \rho \tag{12}$$

where ρ is the charge density. The confusion in much of the literature arises from the fact that while it is true that we consider a plasma of overall neutrality ($\overline{\rho} = 0$, i.e. the charge density of the free electrons is balanced by a uniform positive background charge, for example), the equation obtained from (12) by substituting $\rho = 0$, i.e.

$$\nabla \cdot \vec{D} = 0$$

or

$$\nabla \cdot \epsilon\vec{E} = 0 \;, \tag{13}$$

which has been used by Wyatt[1] and others, is <u>not correct</u> for the case

of time-dependent fields. The correct equation must be obtained on a more igorous basis. To do this, we proceed from the more fundamental equation of charge conservation, which may be written in the form

$$\nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0 \tag{14}$$

If we now substitute Eqs. (5) and (12) and recall that all quantities have the time variation $e^{-i\omega t}$, Eq. (14) becomes

$$\nabla \cdot (\sigma - i\omega\epsilon)\vec{E} = 0 \tag{15}$$

This divergence equation is the correct equation, replacing the <u>incorrect</u> equation, (13), for the case of time-dependent fields. We note from Eq. (9), that Eq. (15) can be rewritten in the form

$$\nabla \cdot (-k_1 \vec{E}) = 0$$

Consequently, we see that it follows automatically from Eq. (7) and thus does not represent an independent equation.

Our problem is thus completely defined by the two Maxwell equations, (6) and (7); the two divergence equations (11 and 15) are not independent equations, and therefore are irrelevant.

III. REDUCTION OF MAXWELL'S EQUATIONS

We now address ourselves to the problem of solving the Maxwell equations, (6) and (7), for the special case of a spherically symmetric medium, i.e. one for which ϵ and σ (and hence k_1 and k) are functions only of the radial distance r from the origin. We further assume that the spherically symmetric medium is finite in extent, with outer radius $r = b$.

The derivation which follows parallels closely that of Born and Wolf[4] for the homogeneous case; we also make use of the notation of Wyatt.[1] Because of the spherical symmetry of the problem, it is most convenient to use spherical coordinates (r, θ, φ). Equations (6) and (7) may then be written in component form as follows:

$$k_2 H_r = \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial\theta} (r E_\varphi \sin\theta) - \frac{\partial}{\partial\varphi} (r E_\theta) \right] \tag{16}$$

$$k_2 H_\theta = \frac{1}{r \sin\theta} \left[\frac{\partial E_r}{\partial\varphi} - \frac{\partial}{\partial r} (r E_\varphi \sin\theta) \right] \tag{17}$$

$$k_2 H_\varphi = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rE_\theta\right) - \frac{\partial E_r}{\partial \theta}\right] \tag{18}$$

$$-k_1 E_r = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial \theta}\left(rH_\varphi \sin\theta\right) - \frac{\partial}{\partial \varphi}\left(rH_\theta\right)\right] \tag{19}$$

$$-k_1 E_\theta = \frac{1}{r \sin\theta}\left[\frac{\partial H_r}{\partial \varphi} - \frac{\partial}{\partial r}\left(rH_\varphi \sin\theta\right)\right] \tag{20}$$

$$-k_1 E_\varphi = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rH_\theta\right) - \frac{\partial H_r}{\partial \theta}\right] \tag{21}$$

(In the equations which follow, we shall suppress the factor $e^{-i\omega t}$, common to all field quantities).

We must now find the general solution of the system of partial differential equations, (16)-(21). This general solution may be written as the superposition of two linearly independent field solutions: the so-called transverse magnetic fields $({}^e\vec{E}, {}^e\vec{H})$ for which $H_r \equiv 0$, and the transverse electric fields $({}^m\vec{E}, {}^m\vec{H})$, for which $E_r \equiv 0$.

Turning our attention first to the case of the transverse magnetic fields $({}^e\vec{E}, {}^e\vec{H})$, Eqs. (16)-(21) take the form

$$\frac{\partial}{\partial \theta}\left(r\, {}^eE_\varphi \sin\theta\right) - \frac{\partial}{\partial \varphi}\left(r\, {}^eE_\theta\right) = 0 \tag{22}$$

$$k_2\,{}^eH_\theta = \frac{1}{r\sin\theta}\left[\frac{\partial\,{}^eE_r}{\partial\varphi} - \frac{\partial}{\partial r}\left(r\,{}^eE_\varphi \sin\theta\right)\right] \tag{23}$$

$$k_2\,{}^eH_\varphi = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\,{}^eE_\theta\right) - \frac{\partial\,{}^eE_r}{\partial\theta}\right] \tag{24}$$

$$-k_1\,{}^eE_r = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(r\,{}^eH_\varphi \sin\theta\right) - \frac{\partial}{\partial\varphi}\left(r\,{}^eH_\theta\right)\right] \tag{25}$$

$$k_1\,{}^eE_\theta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,{}^eH_\varphi\right) \tag{26}$$

$$-k_1\,{}^eE_\varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,{}^eH_\theta\right) \tag{27}$$

Our problem now is to find expressions for ${}^e\vec{E}$, ${}^e\vec{H}$ which satisfy all of the Eqs. (22) through (27). This set of equations may be modified by substituting Eqs. (26) and (27) into (23) and (24), which may then be written in the form (after multiplication by k_1):

$$\left[k_1\frac{\partial}{\partial r}\left(\frac{1}{k_1}\frac{\partial}{\partial r}\right) + k^2\right] r\,{}^eH_\theta = \frac{-k_1}{\sin\theta}\,\frac{\partial\,{}^eE_r}{\partial\varphi} \tag{28}$$

$$\left[k_1\frac{\partial}{\partial r}\left(\frac{1}{k_1}\frac{\partial}{\partial r}\right) + k^2\right] r\,{}^eH_\varphi = k_1\,\frac{\partial\,{}^eE_r}{\partial\theta} \tag{29}$$

Equation (22) may be satisfied identically by choosing ${}^{e}E_{\varphi}$ and ${}^{e}E_{\theta}$ to be given as the gradient of a scalar function ${}^{e}U$:

$$
\begin{aligned}
{}^{e}E_{\varphi} &= \frac{1}{r \sin\theta} \frac{\partial\, {}^{e}U}{\partial\varphi} \\
{}^{e}E_{\theta} &= \frac{1}{r} \frac{\partial\, {}^{e}U}{\partial\theta}
\end{aligned}
\tag{30}
$$

Moreover, if we express ${}^{e}U$ in terms of another scalar function ${}^{m}\Omega$ as follows:

$$
{}^{e}U = \frac{1}{k_1} \frac{\partial}{\partial r} \left(r\, {}^{e}\Omega \right) \quad , \tag{31}
$$

Eq. (30) yields the following expressions for ${}^{e}E_{\varphi}$ and ${}^{e}E_{\theta}$ in terms of ${}^{e}\Omega$:

$$
{}^{e}E_{\varphi} = \frac{1}{k_1 r \sin\theta} \frac{\partial^2}{\partial r\, \partial\varphi} \left(r\, {}^{e}\Omega \right) \tag{32}
$$

$$
{}^{e}E_{\theta} = \frac{1}{k_1 r} \frac{\partial^2}{\partial r\, \partial\theta} \left(r\, {}^{e}\Omega \right) \tag{33}
$$

Substituting these into Eqs. (26) and (27), the latter become

$$
\frac{\partial^2}{\partial r\, \partial\theta} \left(r\, {}^{e}\Omega \right) = \frac{\partial}{\partial r} \left(r\, {}^{e}H_{\varphi} \right) \tag{34}
$$

$$\frac{1}{\sin\theta}\frac{\partial^2}{\partial r\,\partial\varphi}\left(r\ {}^e\Omega\right) = -\frac{\partial}{\partial r}\left(r\ {}^eH_\theta\right) \tag{35}$$

These in turn may be satisfied identically if ${}^eH_\varphi$ and ${}^eH_\theta$ are expressed in terms of ${}^e\Omega$ as follows:

$$ {}^eH_\varphi = \frac{\partial\,{}^e\Omega}{\partial\theta} \tag{36}$$

$$ {}^eH_\theta = -\frac{1}{\sin\theta}\frac{\partial\,{}^e\Omega}{\partial\varphi} \tag{37}$$

Of the original system of equations, we have now satisfied Eqs. (22), (26) and (27), in the course of which we have obtained expressions (32), (33), (36) and (37) for the four angular field components of ${}^e\vec{E}$ and ${}^e\vec{H}$ in terms of a scalar function ${}^e\Omega$. We must yet satisfy Eqs. (25) and (28) through (29)(which are equivalent to 23 and 24). To begin, we substitute expressions (36) and (37) into (25), obtaining

$$-k_1\ {}^eE_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\,{}^e\Omega}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^{2}\,{}^e\Omega}{\partial\varphi^2}\right] \tag{38}$$

If we now substitute Eq. (38), as well as (36) and (37), into Eqs. (28) and (29), these take the form

$$\frac{\partial}{\partial \varphi}\left\{\left[k_1 \frac{\partial}{\partial r}\left(\frac{1}{k_1}\frac{\partial}{\partial r}\right) + k^2\right] r\, {}^e\Omega + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial\, {}^e\Omega}{\partial \theta}\right) + \frac{1}{r \sin^2\theta}\frac{\partial^{2}\, {}^e\Omega}{\partial \varphi^2}\right\} = 0 \tag{39}$$

$$\frac{\partial}{\partial \theta}\left\{\left[k_1 \frac{\partial}{\partial r}\left(\frac{1}{k_1}\frac{\partial}{\partial r}\right) + k^2\right] r\, {}^e\Omega + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial\, {}^e\Omega}{\partial \theta}\right) + \frac{1}{r \sin^2\theta}\frac{\partial^{2}\, {}^e\Omega}{\partial \varphi^2}\right\} = 0 \tag{40}$$

These equations state that the partial derivatives with respect to φ and θ of one and the same expression vanish simultaneously. Both equations may be satisfied simultaneously by assuming that the bracket itself vanishes,* i.e.,

$$\left[k_1 \frac{\partial}{\partial r}\left(\frac{1}{k_1}\frac{\partial}{\partial r}\right) + k^2\right] r\, {}^e\Omega + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial\, {}^e\Omega}{\partial \theta}\right) + \frac{1}{r \sin^2\theta}\frac{\partial^{2}\, {}^e\Omega}{\partial \varphi^2} = 0 \tag{41}$$

---

* This is a <u>sufficient</u> but not a <u>necessary</u> condition. The more general condition, both necessary and sufficient, allows the bracket to be an arbitrary function of r. This in turn would entail a modification in expression (43) for E_r given below. Here we shall content ourselves with the narrower condition (41). The additional degree of freedom provided by the more general condition is somewhat in the nature of a gage transformation.

Equation (41) represents an equation for the scalar function ${}^{e}\Omega$. It may be rewritten in the more concise form

$$\nabla^2\, {}^{e}\Omega - \frac{1}{k_1 r}\frac{\partial k_1}{\partial r}\frac{\partial}{\partial r}\left(r\, {}^{e}\Omega\right) + k^2\, {}^{e}\Omega = 0 \tag{42}$$

Finally, by substituting Eq. (41) into Eq. (38), the latter becomes an equation for ${}^{e}E_r$ in terms of ${}^{e}\Omega$, to wit:

$${}^{e}E_r = \frac{1}{k_1}\left[\frac{\partial^2}{\partial r^2} - \frac{1}{k_1}\frac{\partial k_1}{\partial r}\frac{\partial}{\partial r} + k^2\right] r\, {}^{e}\Omega \tag{43}$$

We have thus succeeded in satisfying all of the original equations (Eqs. (22) through (27), or equations derived from them), in the course of which we have obtained expressions for all field quantities in terms of a single scalar function ${}^{e}\Omega$ which must satisfy Eq. (42).

We now turn our attention to the case of transverse electric fields (${}^{m}\vec{E}$, ${}^{m}\vec{H}$), characterized by the condition ${}^{m}E_r \equiv 0$. The system of equations (16) through (21) then takes the form:

$$k_2\, {}^{m}H_r = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}\left(r\, {}^{m}E_{\varphi} \sin\theta\right) - \frac{\partial}{\partial\varphi}\left(r\, {}^{m}E_{\theta}\right)\right] \tag{44}$$

$$k_2\,{}^mH_\theta = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\,{}^mE_\varphi\right) \tag{45}$$

$$k_2\,{}^mH_\varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,{}^mE_\theta\right) \tag{46}$$

$$\frac{\partial}{\partial\theta}\left(r\,{}^mH_\varphi \sin\theta\right) - \frac{\partial}{\partial\varphi}\left(r\,{}^mH_\theta\right) = 0 \tag{47}$$

$$-k_1\,{}^mE_\theta = \frac{1}{r\sin\theta}\left[\frac{\partial\,{}^mH_r}{\partial\varphi} - \frac{\partial}{\partial r}\left(r\,{}^mH_\varphi\sin\theta\right)\right] \tag{48}$$

$$-k_1\,{}^mE_\varphi = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\,{}^mH_\theta\right) - \frac{\partial\,{}^mH_r}{\partial\theta}\right] \tag{49}$$

We observe that this system of equations may be obtained from the previous system (Eqs. (22) through (27)) for the transverse magnetic fields simply by the transformations:

$${}^e\vec{E} \rightarrow {}^m\vec{H}\,,\ {}^e\vec{H} \rightarrow -{}^m\vec{E}\,,\ k_1 \rightarrow k_2\,,\ k_2 \rightarrow k_1 \tag{50}$$

Inasmuch as the transformations in (50) convert the system of Equations (44) through (49) identically into our previous system of Eqs. (22) through (27), all of the results obtained above for the previous system may be transformed by means of Eq. (50) to apply to the new system

of equations for the transverse electric fields. Accordingly, we may define a scalar function ${}^{m}\Omega$, in terms of which the transverse electric field components are obtained by transforming Eqs. (32), (33), (36), (37) and (43), which yields

$$\left.\begin{aligned}
{}^{m}H_{r} &= \frac{1}{k_2}\left[\frac{\partial^2}{\partial r^2} + k^2\right] r\,{}^{m}\Omega \\
{}^{m}H_{\theta} &= \frac{1}{k_2 r}\,\frac{\partial^2}{\partial r\,\partial\theta}\left(r\,{}^{m}\Omega\right) \\
{}^{m}H_{\varphi} &= \frac{1}{k_2 r \sin\theta}\,\frac{\partial^2}{\partial r\,\partial\varphi}\left(r\,{}^{m}\Omega\right) \\
{}^{m}E_{\theta} &= \frac{1}{\sin\theta}\,\frac{\partial\,{}^{m}\Omega}{\partial\varphi} \\
{}^{m}E_{\varphi} &= -\,\frac{\partial\,{}^{m}\Omega}{\partial\theta}
\end{aligned}\right\} \tag{51}$$

where the equation satisfied by ${}^{m}\Omega$ is obtained by transforming Eq. (42) which becomes

$$\nabla^2\,{}^{m}\Omega + k^2\,{}^{m}\Omega = 0 \tag{52}$$

Note that in transforming Eqs. (42) and (43), the term $\partial k_1/\partial r$ becomes $\partial k_2/\partial r$, which vanishes since k_2 is constant.

Finally, if we assume that ${}^e\Omega$ and ${}^m\Omega$ represent the most general solutions of Eqs. (42) and (52), respectively, the total electromagnetic fields $(\vec{E} = {}^e\vec{E} + {}^m\vec{E}\,,\ \vec{H} = {}^e\vec{H} + {}^m\vec{H})$ in the medium may be obtained by combining Eqs. (32), (33), (36), (37), (43) with (51) which yields

$$E_r = \frac{1}{k_1}\left[\frac{\partial^2}{\partial r^2} - \frac{1}{k_1}\frac{\partial k_1}{\partial r}\frac{\partial}{\partial r} + k^2\right] r\,{}^e\Omega \tag{53}$$

$$E_\theta = \frac{1}{k_1 r}\frac{\partial^2}{\partial r\,\partial\theta}\left(r\,{}^e\Omega\right) + \frac{1}{\sin\theta}\frac{\partial\,{}^m\Omega}{\partial\varphi} \tag{54}$$

$$E_\varphi = \frac{1}{k_1 r\sin\theta}\frac{\partial^2}{\partial r\,\partial\varphi}\left(r\,{}^e\Omega\right) - \frac{\partial\,{}^m\Omega}{\partial\theta} \tag{55}$$

$$H_r = \frac{1}{k_2}\left[\frac{\partial^2}{\partial r^2} + k^2\right] r\,{}^m\Omega \tag{56}$$

$$H_\theta = -\frac{1}{\sin\theta}\frac{\partial\,{}^e\Omega}{\partial\varphi} + \frac{1}{k_2 r}\frac{\partial^2}{\partial r\,\partial\theta}\left(r\,{}^m\Omega\right) \tag{57}$$

$$H_\varphi = \frac{\partial\,{}^e\Omega}{\partial\theta} + \frac{1}{k_2 r\sin\theta}\frac{\partial^2}{\partial r\,\partial\varphi}\left(r\,{}^m\Omega\right) \tag{58}$$

The two scalar functions ${}^e\Omega$ and ${}^m\Omega$ entering into expressions (53) through (58) must be obtained by solving Eqs. (42) and (52).

Inasmuch as we have assumed that k_1 depends only on r, both equations may be separated in spherical coordinates according to standard methods, and the general solutions may be written in the forms:

$$r\,{}^{e}\Omega = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} W_\ell(r)\left[a_m \cos m\varphi + b_m \sin m\varphi\right] P_\ell^m(\cos\theta) \qquad (59)$$

$$r\,{}^{m}\Omega = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} G_\ell(r)\left[c_m \cos m\varphi + d_m \sin m\varphi\right] P_\ell^m(\cos\theta) \qquad (60)$$

where $P_\ell^m(\cos\theta)$ is the associated Legendre polynomial,[4] a_m, b_m, c_m, and d_m are arbitrary constants, and where $W_\ell(r)$ and $G_\ell(r)$ are the general solutions of the ordinary differential equations

$$\frac{d^2W_\ell}{dr^2} - \frac{1}{k_1}\frac{dk_1}{dr}\frac{dW_\ell}{dr} + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right] W_\ell = 0 \qquad (61)$$

$$\frac{d^2G_\ell}{dr^2} + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right] G_\ell = 0 \qquad (62)$$

Recalling that $k_1 k_2 = -k^2$ and that k_2 is constant, Eq. (62) may be rewritten in the form

$$\frac{d^2 W_\ell}{dr^2} - \frac{2}{k}\frac{dk}{dr}\frac{dW_\ell}{dr} + \left[\kappa^2 - \frac{\ell(\ell+1)}{r^2}\right] W_\ell = 0 \tag{63}$$

We note that Eqs. (42) and (52) (which lead to Eqs. (61) and (62), respectively) were obtained as a direct consequence of the two basic Maxwell equations, (6) and (7); they do not arise from the requirements $\nabla \cdot \vec{H} = 0$, $\nabla \cdot \vec{D} = 0$, as stated by Wyatt[(1)]; for example, the latter equation is even incorrect, as we have seen above.

This completes the reduction of Maxwell's equations. Given a specific dependence $k(r)$, the only remaining problem is to find the general solutions W_ℓ and G_ℓ, and then to determine the arbitrary constants in expressions (59) and (60) (including those implicit in W_ℓ and G_ℓ) by imposing suitable boundary conditions.

IV. THE FIELD OF THE INCIDENT PLANE WAVE

In the present report, we are interested in the problem of the scattering of a plane wave by spherically symmetric media such as described above. We shall assume that the inhomogeneous medium is finite in extent, with outer radius $r = b$. Moreover, we shall assume that the incident plane wave is linearly polarized. The coordinate system may then be chosen such that the incident wave propagates in the positive z-direction and has its electric field in the x-direction. The fields of the incident wave are then given by

$$\left.\begin{aligned} E_x^i &= e^{i(k^I z - \omega t)} \\ E_y^i &= E_z^i = 0 \\ H_y^i &= \frac{k^I}{\mu_o \omega} e^{i(k^I z - \omega t)} \\ H_x^i &= H_z^i = 0 \end{aligned}\right\} \qquad (64)$$

Here k^I is the constant propagation constant of the outside medium, and we have assumed the electric field to be of unit magnitude.

Inasmuch as it will be necessary later to match boundary conditions at a spherical surface, we must reexpress these fields in spherical coordinates. For our purposes, we shall only require the r-components of both fields, which are easily shown to be

$$E_r^i = e^{ik^I r \cos\theta} \sin\theta \cos\varphi \tag{65}$$

$$H_r^i = \frac{k^I}{\mu_o \omega} e^{ik^I r \cos\theta} \sin\theta \sin\varphi \tag{66}$$

Since it is much more convenient to apply boundary conditions to potentials than to the field components themselves, it behooves us to find the electric and magnetic potential functions ${}^e\Omega^i$, ${}^m\Omega^i$ from which the above fields may be derived. To do this, we proceed from the well-known expansion

$$e^{ik^I r \cos\theta} = \sum_{\ell=1}^{\infty} i^{\ell}(2\ell+1)\, j_\ell(k^I r)\, P_\ell(\cos\theta) \tag{67}$$

where

$$j_\ell(kr) = \sqrt{\frac{\pi}{2k^I r}}\; J_{\ell+1/2}(k^I r) \tag{68}$$

Inasmuch as

$$e^{ik^I r\cos\theta} = \frac{1}{ik^I r}\,\frac{\partial e^{ik^I r\cos\theta}}{\partial(\cos\theta)} \tag{69}$$

and

$$\sin\theta\,\frac{dP_\ell(\cos\theta)}{d(\cos\theta)} = P_\ell^1(\cos\theta)\quad, \tag{70}$$

Eq. (67) may be rewritten in the form

$$e^{ik^I r\cos\theta} = \frac{(\sin\theta)^{-1}}{k^I r}\sum_{\ell=1}^{\infty} i^{\ell-1}(2\ell+1)\,j_\ell(k^I r)\,P_\ell^1(\cos\theta) \tag{71}$$

If we further introduce the new functions

$$\psi_\ell(k^I r) = k^I r\,j_\ell(k^I r) = \sqrt{\frac{\pi k^I r}{2}}\,J_{\ell+1/2}(k^I r)\quad, \tag{72}$$

we find that Eqs. (65) and (66) may be written in the expanded form

$$E_r^i = \frac{\cos\varphi}{k^{I^2} r^2}\sum_{\ell=1}^{\infty} i^{\ell-1}(2\ell+1)\,\psi_\ell(k^I r)\,P_\ell^1(\cos\theta) \tag{73}$$

$$H_r^i = \frac{\sin\varphi}{\mu_o\omega k^I r^2}\sum_{\ell=1}^{\infty} i^{\ell-1}(2\ell+1)\,\psi_\ell(k^I r)\,P_\ell^1(\cos\theta) \tag{74}$$

We also note that the so-called Ricatti-Bessel functions $\psi_\ell(k^I r)$ are solutions of the equation

$$\frac{d^2\psi_\ell}{dr^2} + \left[k^{I^2} - \frac{\ell(\ell+1)}{r^2} \right] \psi_\ell = 0 \tag{75}$$

On the other hand, as shown in the preceding section for a general medium, the fields may also be derived from potentials of the form (59) and (60) by means of the relations (53) and (56). For the case of constant $k = k^I$, Eqs. (62) and (63) for G_ℓ and W_ℓ are both identical with Eq. (75); consequently, we may write

$$G_\ell(r) = W_\ell(r) = \psi_\ell(k^I r) \tag{76}$$

Moreover, we may note from Eqs. (53) and (56) that only the $m=1$ terms in the general expansions (59) and (60) are required to obtain the incident fields (73) and (74). Consequently, we may write the electric and magnetic potentials of the incident plane wave in the form

$$r\,{}^e\Omega^i = \cos\varphi \sum_{\ell=1}^{\infty} a_\ell\, \psi_\ell(k^I r)\, P_\ell^1(\cos\theta) \tag{77}$$

$$r\,{}^m\Omega^i = \sin\varphi \sum_{\ell=1}^{\infty} b_\ell\, \psi_\ell(k^I r)\, P_\ell^1(\cos\theta) \tag{78}$$

The coefficients a_ℓ may be determined by substituting Eq. (77) into (53) which yields

$$E_r^{\,i} = \frac{\cos\varphi}{k_1^I} \sum_{\ell=1}^{\infty} a_\ell \, P_\ell^{\,\prime}(\cos\theta) \left[\frac{d^2\psi_\ell}{dr^2} + k^{I^2}\psi_\ell \right] \tag{79}$$

In view of Eq. (75), this may be rewritten in the form

$$E_r^{\,i} = \frac{\cos\varphi}{k_1^I r^2} \sum_{\ell=1}^{\infty} a_\ell \, \ell(\ell+1)\, \psi_\ell(k^I r)\, P_\ell^{1}(\cos\theta) \tag{80}$$

By comparing Eq. (80) with (73), we then find that the coefficients a_ℓ are given by

$$a_\ell = \frac{k_1^I}{k^{I^2}} \frac{i^{\ell-1}(2\ell+1)}{\ell(\ell+1)} = -\frac{1}{k_2^I} \frac{i^{\ell-1}(2\ell+1)}{\ell(\ell+1)} \tag{81}$$

such that the electric potential (77) becomes

$$r\,{}^e\Omega^i = -\frac{\cos\varphi}{k_2^I} \sum_{\ell=1}^{\infty} \frac{i^{\ell-1}(2\ell+1)}{\ell(\ell+1)} \psi_\ell(k^I r)\, P_\ell^{1}(\cos\theta) \tag{82}$$

In an exactly analogous manner, the magnetic potential of the incident wave is found to be

$$r\,{}^{m}\Omega^{i} = \frac{i \sin\varphi}{k^{I}} \sum_{\ell=1}^{\infty} \frac{i^{\ell-1}(2\ell+1)}{\ell(\ell+1)}\, \psi_{\ell}(k^{I} r)\, P_{\ell}^{1}(\cos\theta) \qquad (83)$$

V. SCATTERING COEFFICIENTS FOR A SPHERE

A. General Case of an Inhomogeneous Sphere

We now turn to the general problem of the scattering of the plane wave described in Section IV by a spherically symmetric inhomogeneous medium of radius $r=b$. We shall call the region outside the sphere $(r > b)$ Region I, that inside the sphere Region II. The problem is solved by finding the general solution for the fields everywhere and then imposing the required boundary conditions.

The general field in Region I is composed of the field of the incident plane wave plus the scattered field. The potentials of the former are given by Eqs. (82) and (83). The potentials of the scattered field are given by the general expression (59) and (60), where $W_\ell(r)$ and $G_\ell(r)$ are both solutions of the equation ($k = k^I$ is constant in Region I).

$$\frac{d^2 u_\ell}{dr^2} + \left[k^{I^2} - \frac{\ell(\ell+1)}{r^2} \right] u_\ell = 0 \tag{84}$$

Equation (84) has two linearly independent solutions (one of these is $\psi_\ell(k^I r)$, as we saw previously). For the scattered wave, we must choose that linear combination which for large r behaves asymptotically as $e^{ik^I r}/r$. This solution is easily seen to be

$$W_\ell(r) = G_\ell(r) = \alpha_\ell \zeta_\ell^{(1)}(k^I r) = \alpha_\ell \sqrt{\frac{\pi k^I r}{2}}\, H^{(1)}_{\ell+1/2}(k^I r) , \qquad (85)$$

where α_ℓ are arbitr-·y constants.

Moreover, inasmuch as we must satisfy boundary conditions at $r=b$ for all values of θ and φ, and the potentials ${}^e\Omega^i$, ${}^m\Omega^i$ involve only $\cos\varphi$ and $\sin\varphi$, respectively, it is evident that the same must be true of the potentials of the scattered field as well as of the field transmitted into Region II.

Accordingly, the potentials of the scattered field in Region I may be conveniently written in the form

$$r\,{}^e\Omega^s = -\frac{\cos\varphi}{k^I_2} \sum_{\ell=1}^{\infty} {}^eB_\ell \frac{i^{\ell-1}(2\ell+1)}{\ell(\ell+1)} \zeta_\ell^{(1)}(k^I r)\, P_\ell^1(\cos\theta) \qquad (86)$$

$$r\,{}^m\Omega^s = \frac{i\sin\varphi}{k^I} \sum_{\ell=1}^{\infty} {}^mB_\ell \frac{i^{\ell-1}(2\ell+1)}{\ell(\ell+1)} \zeta_\ell^{(1)}(k^I r)\, P_\ell^1(\cos\theta) \qquad (87)$$

where the arbitrary constants ${}^{e}B_{\ell}$ and ${}^{m}B_{\ell}$ have been defined as above in order to facilitate the later matching of boundary conditions. Similarly, the potentials of the fields in Region II can be written in the form

$$r\,{}^{e}\Omega^{II} = -\frac{\cos\varphi}{k_2^{I}} \sum_{\ell=1}^{\infty} {}^{e}A_{\ell} \frac{i^{\ell-1}(2\ell+1)}{\ell(\ell+1)} W_{\ell}(r)\, P_{\ell}^{1}(\cos\theta) \qquad (88)$$

$$r\,{}^{m}\Omega^{II} = \frac{i\sin\varphi}{k^{I}} \sum_{\ell=1}^{\infty} {}^{m}A_{\ell} \frac{i^{\ell-1}(2\ell+1)}{\ell(\ell+1)} G_{\ell}(r)\, P_{\ell}^{1}(\cos\theta) \quad , \qquad (89)$$

where in this case $W_{\ell}(r)$ and $G_{\ell}(r)$ represent particular (rather than general) solutions of Eqs. (62) and (63), specifically those solutions having no singularities at the origin.

We now turn to the boundary conditions which must be satisfied at the boundary $r=b$. These are

$$\left.\begin{aligned} E_{\theta}^{I} &= E_{\theta}^{II} \;, \quad E_{\varphi}^{I} = E_{\varphi}^{II} \\ H_{\theta}^{I} &= H_{\theta}^{II} \;, \quad H_{\varphi}^{I} = H_{\varphi}^{II} \end{aligned} \;\; ; \quad r = b \right\} \qquad (90)$$

We wish to express these boundary conditions in terms of the potential functions ${}^{e}\Omega$ and ${}^{m}\Omega$. To do this, we observe from Eqs. (54), (55), (57) and (58), that Eqs. (90) are satisfied, if we have

$$\frac{1}{k_1^{\,I}} \frac{\partial}{\partial r} (r \, {}^e\Omega^I) = \frac{1}{k_1^{\,II}(b)} \frac{\partial}{\partial r} (r \, {}^e\Omega^{II}) \quad , \quad r = b \tag{91}$$

$${}^m\Omega^I = {}^m\Omega^{II} \quad , \quad r = b \tag{92}$$

$${}^e\Omega^I = {}^e\Omega^{II} \quad , \quad r = b \tag{93}$$

$$\frac{\partial}{\partial r} (r \, {}^m\Omega^I) = \frac{\partial}{\partial r} (r \, {}^m\Omega^{II}) \quad , \quad r = b \tag{94}$$

where Ω^I represents the total potential in Region I, $\Omega^I = \Omega^i + \Omega^s$. Eqs. (91) through (94) represent four simultaneous equations for the four unknown coefficients ${}^eA_\ell$, ${}^mA_\ell$, ${}^eB_\ell$, ${}^mB_\ell$.

At this point, it is convenient to introduce a new variable ρ defined by

$$\rho = k^I r \tag{95}$$

Since k^I is a constant, the boundary conditions, Eqs. (91) through (94), can be rewritten in the form

$$\frac{1}{k_1^{\,I}} \frac{\partial}{\partial \rho} (\rho \, {}^e\Omega^I) = \frac{1}{k_1^{\,II}(b)} \frac{\partial}{\partial \rho} (\rho \, {}^e\Omega^{II}) \quad , \quad \rho = x \tag{96}$$

$${}^m\Omega^I = {}^m\Omega^{II} \quad , \quad \rho = x \tag{97}$$

$$ {}^{e}\Omega^{I} = {}^{e}\Omega^{II} \quad , \quad \rho = x \tag{98} $$

$$ \frac{\partial}{\partial \rho} (\rho\, {}^{m}\Omega^{I}) = \frac{\partial}{\partial \rho} (\rho\, {}^{m}\Omega^{II}) \quad , \quad \rho = x \tag{99} $$

where we have defined

$$ x = \rho(b) = k^{I} b \tag{100} $$

The Ricatti-Bessel functions entering into the potentials (82) and (83) of the incident field and into the potentials (86) and (87) of the scattered field already have the argument ρ. We may also consider the functions W_ℓ and G_ℓ occurring in the potentials (88) and (89) as being functions of ρ, and write

$$ r\, {}^{e}\Omega^{II} = - \frac{\cos\varphi}{k_2^{I}} \sum_{\ell=1}^{\infty} {}^{e}A_\ell \frac{i^{\ell-1}(2\ell+1)}{\ell(\ell+1)} W_\ell(\rho)\, P_\ell^{1}(\cos\theta) \tag{101} $$

$$ r\, {}^{m}\Omega^{II} = \frac{i \sin\varphi}{k^{I}} \sum_{\ell=1}^{\infty} {}^{m}A_\ell \frac{i^{\ell-1}(2\ell+1)}{\ell(\ell+1)} G_\ell(\rho)\, P_\ell^{1}(\cos\theta) \tag{102} $$

The differential equations satisfied by $W_\ell(\rho)$ and $G_\ell(\rho)$ are obtained from Eqs. (62) and (63) by substituting $\rho = k^{I} r$, and we find

$$\frac{d^2 W_\ell}{d\rho^2} - \frac{2}{n}\frac{dn}{d\rho}\frac{dW_\ell}{d\rho} + \left[n^2 - \frac{\ell(\ell+1)}{\rho^2} \right] W_\ell = 0 \tag{103}$$

$$\frac{d^2 G_\ell}{d\rho^2} + \left[n^2 - \frac{\ell(\ell+1)}{\rho^2} \right] G_\ell = 0 \tag{104}$$

where the relative "refractive index" n corresponding to a medium with propagation constant k is given by

$$n = \frac{k}{k^I} \quad , \quad \text{e.g.} \quad n_2 = \frac{k^{II}}{k^I} \tag{105}$$

We now substitute the potentials (82) through (83), (86) through (87), and (101) through (102) into the boundary conditions (96) through (99), obtaining the following four equations for the coefficients ${}^eA_\ell$, ${}^mA_\ell$, ${}^eB_\ell$, ${}^mB_\ell$:

$${}^eB_\ell \, \zeta_\ell^{(1)}(x) - {}^eA_\ell \, W_\ell(x) = - \psi_\ell(x) \tag{106}$$

$${}^mB_\ell \, \zeta_\ell^{(1)}(x) - {}^mA_\ell \, G_\ell(x) = - \psi_\ell(x) \tag{107}$$

$${}^eB_\ell \, \frac{\zeta_\ell^{(1)'}(x)}{k_1^I} - {}^eA_\ell \, \frac{W_\ell'(x)}{k_1^{II}(b)} = - \frac{\psi_\ell'(x)}{k_1^I} \tag{108}$$

$$ {}^{m}B_{\ell}\, \zeta_{\ell}^{(1)'}(x) - {}^{m}A_{\ell}\, G_{\ell}'(x) = -\psi_{\ell}'(x) \tag{109} $$

where primes denote differentiation with respect to the argument. Multiplying Eq. (108) by $k_1^{\,I}$ and noting that

$$ \frac{k_1^{\,I}}{k_1^{\,II}(b)} = \frac{k_2\, k_1^{\,I}}{k_2\, k_1^{\,II}(b)} = \frac{k^{I^2}}{\left[k^{II}(b)\right]^2} = \frac{1}{n_2^{\,2}(b)} \tag{110} $$

Eq. (108) may be rewritten in the form

$$ {}^{e}B_{\ell}\, \zeta_{\ell}^{(1)'}(x) - {}^{e}A_{\ell} \frac{W_{\ell}'(x)}{\left[n_2(b)\right]^2} = -\psi_{\ell}'(x) \tag{108'} $$

Only the scattering coefficients ${}^{e}B_{\ell}$, ${}^{m}B_{\ell}$ are required to calculate all of the scattering quantities of interest. The coefficient ${}^{e}B_{\ell}$ is obtained by simultaneously solving Eqs. (106) and (108), which yields

$$ {}^{e}B_{\ell} = \frac{\left[n_2^{\,2}(b)\, \psi_{\ell}'(x) - \dfrac{\psi_{\ell}(x)\, W_{\ell}'(x)}{W_{\ell}(x)} \right]}{\left[\zeta_{\ell}^{(1)}(x) \dfrac{W_{\ell}'(x)}{W_{\ell}(x)} - n_2^{\,2}(b)\, \zeta_{\ell}^{(1)'}(x) \right]} \tag{111} $$

Similarly, ${}^{m}B_{\ell}$ is obtained by solving Eqs. (107) and (108), and is found to be

$$
{}^{m}B_{\ell} = \frac{\left[\psi_{\ell}(x)\,\frac{G_{\ell}'(x)}{G_{\ell}(x)} - \psi_{\ell}'(x)\right]}{\left[\zeta_{\ell}^{(1)'}(x) - \zeta_{\ell}^{(1)}(x)\,\frac{G_{\ell}'(x)}{G_{\ell}(x)}\right]} \tag{112}
$$

The above expressions may be written more compactly by introducing the notations

$$
D_{\ell}(x) = \frac{\psi_{\ell}'(x)}{\psi_{\ell}(x)} \tag{113}
$$

$$
\Gamma_{\ell}(x) = \frac{\zeta_{\ell}^{(1)'}(x)}{\zeta_{\ell}^{(1)}(x)} \tag{114}
$$

$$
\omega_{\ell}(x) = \frac{W_{\ell}'(x)}{W_{\ell}(x)} \tag{115}
$$

$$
\gamma_{\ell}(x) = \frac{G_{\ell}'(x)}{G_{\ell}(x)} \tag{116}
$$

with which the scattering coefficients take the form

$$
{}^{e}B_{\ell} = \frac{\psi_{\ell}(x)}{\zeta_{\ell}^{(1)}(x)} \left[\frac{n_2^{2}(b)\, D_{\ell}(x) - \omega_{\ell}(x)}{\omega_{\ell}(x) - n_2^{2}(b)\, \Gamma_{\ell}(x)}\right] \tag{117}
$$

$$ {}^{m}B_{\ell} = \frac{\psi_{\ell}(x)}{\zeta_{\ell}^{(1)}(x)} \; \frac{\gamma_{\ell}(x) - D_{\ell}(x)}{\Gamma_{\ell}(x) - \gamma_{\ell}(x)} \tag{118} $$

This completes the calculation of the scattering coefficients for the case of an inhomogeneous sphere. (The result obtained for ${}^{m}B_{\ell}$ by Wyatt[1] for the same case is incorrect.) The coefficients ${}^{e}B_{\ell}$ and ${}^{m}B_{\ell}$ may be found explicitly for any given variation $n(r)$ by solving the differential equations (103) and (104). These were solved numerically by Wyatt[1] for a very particular variation of $n(r)$; they may be solved analytically for $n(r) = n_2 =$ constant and for the case where n obeys a power law. The former is discussed in Section V-B, the latter in Section VI-D-2 of the present report.

B. Special Case of the Homogeneous Sphere

We shall now specialize the solution found above to the case of a homogeneous sphere with a constant propagation constant k^{II}, $(k^{II} \neq k^{I})$. We thus also have $n_2(r) = n_2 =$ constant. This corresponds to case (a) in Figure 1. In keeping with the notation of Fig. 1, the radius of the sphere is now $r = a$; accordingly, we define

$$ y = k^{I} a \tag{119} $$

and the scattering coefficients are given by Eqs. (111) through (112) with x replaced by y.

These may be simplified, however, by noting that $W_\ell(\rho)$, $G_\ell(\rho)$ for a homogeneous medium are given by

$$W_\ell(\rho) = G_\ell(\rho) = \psi_\ell(n_2\rho) \tag{120}$$

where the function ψ_ℓ is defined by Eq. (72). If we further note that

$$W_\ell'(\rho) = G_\ell'(\rho) = n_2\,\psi_\ell'(n_2\rho) \tag{121}$$

where primes denote differentiation with respect to the argument, we find from Eqs. (113) through (116) that

$$\omega_\ell(y) = \gamma_\ell(y) = n_2\,D_\ell(n_2 y) \tag{122}$$

Accordingly, the scattering coefficients given by Eqs. (117) through (118) (with x replaced by y) then become

$$^eB_\ell = \frac{\psi_\ell(y)}{\zeta_\ell^{(1)}(y)}\left[\frac{n_2 D_\ell(y) - D_\ell(n_2 y)}{D_\ell(n_2 y) - n_2\Gamma_\ell(y)}\right] \tag{123}$$

$$^mB_\ell = \frac{\psi_\ell(y)}{\zeta_\ell^{(1)}(y)}\left[\frac{n_2 D_\ell(n_2 y) - D_\ell(y)}{\Gamma_\ell(y) - n_2 D_\ell(n_2 y)}\right] \tag{124}$$

These expressions represent a considerable simplification over the corresponding expressions (117) and (118), inasmuch as the only transcendental functions involved in epxressions (123) and (124) are the functions $\psi_\ell(x)$, $\zeta_\ell^{(1)}(x)$; no numerical integration of differential equations is required. The scattering coefficients given by Eqs. (123) and (124) represent the complete solution of case (a) of Figure (1).

VI SCATTERING COEFFICIENTS FOR A SPHERE WITH A CENTRAL CORE

A. General Case

We now turn to the more general case of a sphere with a central homogeneous core, which includes cases (b), (c) and (d) of Figure (1). The situation to be considered is illustrated in Fig. (2) below.

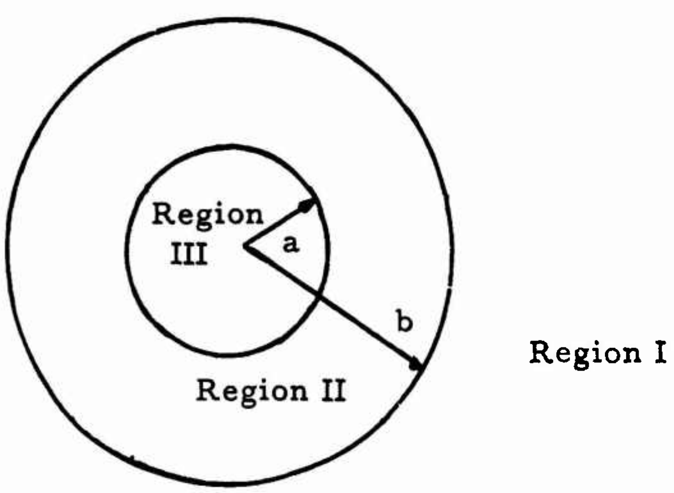

FIGURE 2

Regions I and III are considered to be homogeneous with constant propagation constants k^{I} and k^{III}, respectively, while k^{II} may be

an arbitrary function of r. The purpose of this section is to find expressions for the scattering coefficients for this general case; in subsequent sections, we shall consider particular variations of k^{II} (or $n_2 = k^{II}/k^{I}$) in Region II.

As before, the problem consists of finding the potentials everywhere and then applying the boundary conditions (96) through (99) at both r=a and r=b. The potentials of both the incident and scattered fields in Region I are again given by Eqs. (82) - (83) and (86) - (87), respectively. Similarly, since k^{III} is constant in Region III, the potentials in Region III may be written in the form

$$r\, e_{\Omega}^{III} = -\frac{\cos\varphi}{k_2^{I}} \sum_{\ell=1}^{\infty} c_{\ell} \frac{i^{\ell-1}(2\ell+1)}{\ell(\ell+1)} \psi_{\ell}(n_3\rho)\, P_{\ell}^{1}(\cos\theta) \tag{125}$$

$$r\, m_{\Omega}^{III} = \frac{i\sin\varphi}{k^{I}} \sum_{\ell=1}^{\infty} d_{\ell} \frac{i^{\ell-1}(2\ell+1)}{\ell(\ell+1)} \psi_{\ell}(n_3\rho)\, P_{\ell}^{1}(\cos\theta) \tag{126}$$

Finally, the potentials in Region II may be obtained from the general expressions (59) through (60), where only the m=1 terms can contribute because of the boundary conditions.

As in Section V-A, we shall consider the functions W_ℓ and G_ℓ occurring in Eqs. (59) and (60) to be functions of the argument $\rho = k^I r$; $W_\ell(\rho)$ and $G_\ell(\rho)$ then represent the general solution of Eqs. (103) and (104). Inasmuch as Region II does not contain the origin, Eqs. (103) and (104) will both admit of two linearly independent solutions, so that the general solutions may be written in the form

$$W_\ell(\rho) = \alpha_\ell X_\ell(\rho) + \beta_\ell Y_\ell(\rho) \tag{127}$$

$$G_\ell(\rho) = \gamma_\ell U_\ell(\rho) + \delta_\ell V_\ell(\rho) \tag{128}$$

where α_ℓ, β_ℓ, γ_ℓ, δ_ℓ are arbitrary constants. The potentials in Region II may then be written in the form

$$r\, {}^{e}\Omega^{II} = -\frac{\cos\varphi}{k_2^I} \sum_{\ell=1}^{\infty} \frac{i^{\ell-1}(2\ell+1)}{\ell(\ell+1)} P_\ell^1(\cos\theta)\left[\alpha_\ell X_\ell(\rho) + \beta_\ell Y_\ell(\rho)\right] \tag{129}$$

$$r\, {}^{m}\Omega^{II} = \frac{i \sin\varphi}{k^I} \sum_{\ell=1}^{\infty} \frac{i^{\ell-1}(2\ell+1)}{\ell(\ell+1)} P_\ell^1(\cos\theta)\left[\gamma_\ell U_\ell(\rho) + \delta_\ell V_\ell(\rho)\right] \tag{130}$$

We must now match the boundary conditions (96) through (99) at $r=b$ ($\rho=x$) and $r=a$ ($\rho=y$). Substituting expressions (82)-(83), (86)-(87) and (129)-(130) into the boundary conditions (96) through (99) for $\rho=x$, we obtain the equations

$$\psi_\ell(x) + {}^{e}B_\ell\, \zeta_\ell^{(1)}(x) - \alpha_\ell X_\ell(x) - \beta_\ell Y_\ell(x) = 0 \tag{131}$$

$$\psi_\ell(x) + {}^{m}B_\ell\, \zeta_\ell^{(1)}(x) - \gamma_\ell U_\ell(x) - \delta_\ell V_\ell(x) = 0 \tag{132}$$

$$\frac{1}{k_1^{\,I}} \psi_\ell'(x) + {}^{e}B_\ell \frac{\zeta_\ell^{(1)'}(x)}{k_1^{\,I}} - \frac{1}{k_1^{\,II}(b)} \left[\alpha_\ell X_\ell'(x) + \beta_\ell Y_\ell'(x)\right] = 0 \tag{133}$$

$$\psi_\ell'(x) + {}^{m}B_\ell\, \zeta_\ell^{(1)'}(x) - \gamma_\ell U_\ell'(x) - \delta_\ell V_\ell'(x) = 0 \tag{134}$$

Similarly, substituting Eqs. (129) - (130) and (125) - (126) into the same boundary conditions for $\rho = y$, we are led to the four equations:

$$\alpha_\ell X_\ell(y) + \beta_\ell Y_\ell(y) = c_\ell \psi_\ell(n_3 y) \tag{135}$$

$$\gamma_\ell U_\ell(y) + \delta_\ell V_\ell(y) = d_\ell \psi_\ell(n_3 y) \tag{136}$$

$$\frac{1}{k_1^{\,II}(a)} \left[\alpha_\ell X_\ell'(y) + \beta_\ell Y_\ell'(y)\right] = c_\ell \frac{n_3}{k_1^{\,II}} \psi_\ell'(n_3 y) \tag{137}$$

$$\gamma_\ell U_\ell'(y) + \delta_\ell V_\ell'(y) = d_\ell n_3 \psi_\ell'(n_3 y) \tag{138}$$

Equations (131) through (138) represent eight simultaneous linear equations for the eight coefficients ${}^{e}B_\ell$, ${}^{m}B_\ell$, c_ℓ, d_ℓ, α_ℓ, β_ℓ,

γ_ℓ , δ_ℓ . Fortunately, these separate into two independent sets of four equations each. Thus, the coefficient ${}^eB_\ell$ may be found from Eqs.(131), (133), (135) and (137), which may be rewritten in the form

$$ {}^eB_\ell \, \zeta_\ell^{(1)}(x) - \alpha_\ell X_\ell(x) - \beta_\ell Y_\ell(x) = -\psi_\ell(x) \qquad (139)$$

$$ {}^eB_\ell \, \zeta_\ell^{(1)'}(x) - \alpha_\ell \frac{X_\ell'(x)}{n_2^2(b)} - \beta_\ell \frac{Y_\ell'(x)}{n_2^2(b)} = -\psi_\ell'(x) \qquad (140$$

$$ \alpha_\ell X_\ell(y) + \beta_\ell Y_\ell(y) - c_\ell \psi_\ell(n_3 y) = 0 \qquad (141)$$

$$ \alpha_\ell X_\ell'(y) + \beta_\ell Y_\ell'(y) - c_\ell \frac{n_2^2(a)}{n_3} \psi_\ell'(n_3 y) = 0 \qquad (142)$$

where Eqs. (140) and (142) were obtained by multiplying Eqs. (133) and (137) by k_1^{I} and $k_1^{II}(a)$, respectively, and noting that

$$ \frac{k_1^{I}}{k_1^{II}(b)} = \frac{k_2 k_1^{I}}{k_2 k_1^{II}(b)} = \frac{k^{I^2}}{k^{II^2}(b)} = \frac{1}{n_2^2(b)} \qquad (143)$$

$$ \frac{k_1^{II}(a) n_3}{k_1^{III}} = \frac{n_3 k_2 k_1^{II}(a)}{k_2 k_1^{III}} = \frac{n_3 k^{II^2}(a)}{k^{III^2}} = \frac{n_2^2(a)}{n_3} \qquad (144)$$

Equations (139) through (142) may be solved by standard methods, and the solution for ${}^{e}B_\ell$ may be written in the determinant form

$$
{}^{e}B_\ell = \frac{\Delta_1}{\Delta_2} = \frac{\begin{vmatrix} -\psi_\ell(x) & & \text{other terms as below} & \\ -\psi_\ell'(x) & & & \\ 0 & & & \\ 0 & & & \end{vmatrix}}{\begin{vmatrix} \zeta_\ell^{(1)}(x) & -X_\ell(x) & -Y_\ell(x) & 0 \\ \zeta_\ell^{(1)'}(x) & -\dfrac{1}{n_2^2(b)} X_\ell'(x) & -\dfrac{1}{n_2^2(b)} Y_\ell'(x) & 0 \\ 0 & X_\ell(y) & Y_\ell(y) & -\psi_\ell(n_3 y) \\ 0 & X_\ell'(y) & Y_\ell'(y) & \dfrac{-n_2^2(a)}{n_3} \psi_\ell'(n_3 y) \end{vmatrix}} \tag{145}
$$

The bottom determinant Δ_2 may be evaluated by first multiplying the first row by $-\zeta_\ell^{(1)'}(x) / \zeta_\ell^{(1)}(x)$ and adding it to the second row, which yields

$$\Delta_2 = \zeta_\ell^{(1)}(x) \begin{vmatrix} \left[\dfrac{\zeta_\ell^{(1)'}(x)}{\zeta_\ell^{(1)}(x)} X_\ell(x) - \dfrac{X_\ell'(x)}{n_2^2(b)}\right] & \left[\dfrac{\zeta_\ell^{(1)'}(x)}{\zeta_\ell^{(1)}(x)} Y_\ell(x) - \dfrac{Y_\ell'(x)}{n_2^2(b)}\right] & 0 \\ X_\ell(y) & Y_\ell(y) & -\psi_\ell(n_3 y) \\ X_\ell'(y) & Y_\ell'(y) & -\dfrac{n_2^2(a)}{n_3} \psi_\ell'(n_3 y) \end{vmatrix}$$

(146)

If in the determinant of Eq. (146), we now add $\dfrac{-n_3}{n_2^2(a)} \dfrac{\psi_\ell(n_3 y)}{\psi_\ell'(n_3 y)}$ times the third row to the second row, we obtain

$$\Delta_2 = \frac{-n_2^2(a)}{n_3} \zeta_\ell^{(1)}(x) \psi_\ell'(n_3 y) \begin{vmatrix} \left[\dfrac{\zeta_\ell^{(1)'}(x)}{\zeta_\ell^{(1)}(x)} X_\ell(x) - \dfrac{X_\ell'(x)}{n_2^2(b)}\right] & \left[\dfrac{\zeta_\ell^{(1)'}(x)}{\zeta_\ell^{(1)}(x)} Y_\ell(x) - \dfrac{Y_\ell'(x)}{n_2^2(b)}\right] \\ \left[X_\ell(y) - \dfrac{n_3 \psi_\ell(n_3 y)}{n_2^2(a) \psi_\ell'(n_3 y)} X_\ell'(y)\right] & \left[Y_\ell(y) - \dfrac{n_3 \psi_\ell(n_3 y)}{n_2^2(a) \psi_\ell'(n_3 y)} Y_\ell'(y)\right] \end{vmatrix}$$

(147)

which may be easily expanded. Thus, making use of the notations (113) and (114) introduced previously, we find

$$\Delta_2 = -\frac{\zeta_\ell^{(1)}(x)\,\psi_\ell'(n_3y)}{n_3 n_2^2(b) D_\ell(n_3y)}\Big\{\Big[n_2^2(b)\,\Gamma_\ell(x)\,X_\ell(x) - X_\ell'(x)\Big]\Big[n_2^2(a)\,Y_\ell(y)\,D_\ell(n_3y) - n_3\,Y_\ell'(y)\Big]$$

$$- \Big[n_2^2(b)\,\Gamma_\ell(x)\,Y_\ell(x) - Y_\ell'(x)\Big]\Big[n_2^2(a)\,D_\ell(n_3y)\,X_\ell(y) - n_3\,X_\ell'(y)\Big]\Big\} \qquad (148)$$

The determinant in the numerator of Eq. (145), Δ_2, is identical to Δ_1 except that $\zeta_\ell^{(1)}(x)$ and $\zeta_\ell^{(1)'}(x)$ are replaced by $-\psi_\ell(x)$ and $-\psi_\ell'(x)$, respectively. Consequently, the value of Δ_1 may be written down immediately from Eq. (148), by making the substitutions $\zeta_\ell^{(1)}(x) \longrightarrow -\psi_\ell(x)$, $\zeta_\ell^{(1)'}(x) \longrightarrow -\psi_\ell'(x)$, and accordingly also $\Gamma_\ell(x) \longrightarrow D_\ell(x)$. We then find,

$$\Delta_1 = \frac{\psi_\ell(x)\,\psi_\ell'(n_3y)}{n_3 n_2^2(b)\,D_\ell(n_3y)}\Big\{\Big[n_2^2(b)\,D_\ell(x)\,X_\ell(x) - X_\ell'(x)\Big]\Big[n_2^2(a)\,Y_\ell(y)\,D_\ell(n_3y) - n_3Y_\ell'(y)\Big]$$

$$- \Big[n_2^2(b)\,D_\ell(x)\,Y_\ell(x) - Y_\ell'(x)\Big]\Big[n_2^2(a)\,X_\ell(y)\,D_\ell(n_3y) - n_3X_\ell'(y)\Big]\Big\} \qquad (149)$$

Thus, the desired scattering coefficient ${}^eB_\ell$ becomes

$$
{}^{e}B_{\ell} = \frac{-\psi_{\ell}(x)}{\zeta_{\ell}^{(1)}(x)} \left\{ \frac{\left[n_2^{\,2}(b)D_{\ell}(x)X_{\ell}(x) - X_{\ell}'(x)\right]\left[n_2^{\,2}(a)Y_{\ell}(y)D_{\ell}(n_3y) - n_3Y_{\ell}'(y)\right] - \left[n_2^{\,2}(b)D_{\ell}(x)Y_{\ell}(x) - Y_{\ell}'(x)\right]\left[n_2^{\,2}(a)D_{\ell}(n_3y)X_{\ell}(y) - n_3X_{\ell}'(y)\right]}{\left[n_2^{\,2}(b)\Gamma_{\ell}(x)X_{\ell}(x) - X_{\ell}'(x)\right]\left[n_2^{\,2}(a)Y_{\ell}(y)D_{\ell}(n_3y) - n_3Y_{\ell}'(y)\right] - \left[n_2^{\,2}(b)\Gamma_{\ell}(x)Y_{\ell}(x) - Y_{\ell}'(x)\right]\left[n_2^{\,2}(a)D_{\ell}(n_3y)X_{\ell}(y) - n_3X_{\ell}'(y)\right]} \right\} \tag{150}
$$

Similarly, the coefficient ${}^{m}B_{\ell}$ may be found by simultaneously solving Eqs. (132), (134), (136) and (138). The solution may be written in determinant form as follows

$$
{}^{m}B_{\ell} = \frac{\Delta_3}{\Delta_4} = \frac{\begin{vmatrix} -\psi_{\ell}(x) & & & \\ -\psi_{\ell}'(x) & & \text{remainder as below} & \\ 0 & & & \\ 0 & & & \end{vmatrix}}{\begin{vmatrix} \zeta_{\ell}^{(1)}(x) & -U_{\ell}(x) & -V_{\ell}(x) & 0 \\ \zeta_{\ell}^{(1)\prime}(x) & -U_{\ell}'(x) & -V_{\ell}'(x) & 0 \\ 0 & U_{\ell}(y) & V_{\ell}(y) & -\psi_{\ell}(n_3y) \\ 0 & U_{\ell}'(y) & V_{\ell}'(y) & -n_3\psi_{\ell}'(n_3y) \end{vmatrix}} \tag{151}
$$

The determinants of Eq. (151) may be evaluated by the same methods used for those of Eq. (145). However, it is simpler to note that if in Eq. (145) we make the substitutions

$$X_\ell(x) \to U_\ell(x)\ , \quad X_\ell'(x) \to U_\ell'(x)\ , \quad Y_\ell(x) \to V_\ell(x)\ , \quad Y_\ell'(x) \to V_\ell'(x)\ , \quad n_2^2(a) = n_2^2(b) \to 1\ ,$$

$$n_3 \to \frac{1}{n_3}\ , \tag{152}$$

we exactly reproduce Eq. (151). Consequently, the solution for ${}^mB_\ell$ may be written down immediately from Eq. (150) by making the substitutions (152), and we obtain

$${}^mB_\ell = \frac{-\psi_\ell(x)}{\zeta_\ell^{(1)}(x)}\left\{\frac{\left[D_\ell(x)U_\ell(x) - U_\ell'(x)\right]\left[n_3 D_\ell(n_3 y)V_\ell(y) - V_\ell'(y)\right] - \left[D_\ell(x)V_\ell(x) - V_\ell'(x)\right]\left[n_3 D_\ell(n_3 y)U_\ell(y) - U_\ell'(y)\right]}{\left[\Gamma_\ell(x)U_\ell(x) - U_\ell'(x)\right]\left[n_3 D_\ell(n_3 y)V_\ell(y) - V_\ell'(y)\right] - \left[\Gamma_\ell(x)V_\ell(x) - V_\ell'(x)\right]\left[n_3 D_\ell(n_3 y)U_\ell(y) - U_\ell'(y)\right]}\right\} \tag{153}$$

Finally, we shall rewrite Eqs. (150) and (153) for ${}^eB_\ell$ and ${}^mB_\ell$ in slightly different form, by introducing the notations (analogous to those of Eqs. (113) through (116):

$$\mu_\ell(x) = \frac{X'_\ell(x)}{X_\ell(x)} \quad , \quad \nu_\ell(x) = \frac{Y'_\ell(x)}{Y_\ell(x)} \quad , \quad \xi_\ell(x) = \frac{U'_\ell(x)}{U_\ell(x)} \quad , \quad \eta_\ell(x) = \frac{V'_\ell(x)}{V_\ell(x)} \tag{154}$$

The coefficients ${}^eB_\ell$ and ${}^mB_\ell$ then take the form:

$${}^eB_\ell = \frac{-\psi_\ell(x)}{\zeta_\ell^{(1)}(x)} \left\{ \frac{X_\ell(x)Y_\ell(y)\left[n_2^2(b)D_\ell(x) - \mu_\ell(x)\right]\left[n_2^2(a)D_\ell(n_3y) - n_3\nu_\ell(y)\right] - X_\ell(y)Y_\ell(x)\left[n_2^2(b)D_\ell(x) - \nu_\ell(x)\right]\left[n_2^2(a)D_\ell(n_3y) - n_3\mu_\ell(y)\right]}{X_\ell(x)Y_\ell(y)\left[n_2^2(b)\Gamma_\ell(x) - \mu_\ell(x)\right]\left[n_2^2(a)D_\ell(n_3y) - n_3\nu_\ell(y)\right] - X_\ell(y)Y_\ell(x)\left[n_2^2(b)\Gamma_\ell(x) - \nu_\ell(x)\right]\left[n_2^2(a)D_\ell(n_3y) - n_3\mu_\ell(y)\right]} \right\} \tag{155}$$

$${}^mB_\ell = \frac{-\psi_\ell(x)}{\zeta_\ell^{(1)}(x)} \left\{ \frac{U_\ell(x)V_\ell(y)\left[D_\ell(x) - \xi_\ell(x)\right]\left[n_3D_\ell(n_3y) - \eta_\ell(y)\right] - V_\ell(x)U_\ell(y)\left[D_\ell(x) - \eta_\ell(x)\right]\left[n_3D_\ell(n_3y) - \xi_\ell(y)\right]}{U_\ell(x)V_\ell(y)\left[\Gamma_\ell(x) - \xi_\ell(x)\right]\left[n_3D_\ell(n_3y) - \eta_\ell(y)\right] - V_\ell(x)U_\ell(y)\left[\Gamma_\ell(x) - \eta_\ell(x)\right]\left[n_3D_\ell(n_3y) - \xi_\ell(y)\right]} \right\} \tag{156}$$

Equations (155) and (156) (or alternatively Eqs. (150) and (153)) represent the final solutions for the scattering coefficients for an inhomogeneous sphere with a homogeneous central core. In order to evaluate them for a specific case with a given dependence $n_2(r)$, it is only necessary to solve Eqs. (103) and (104) and thus obtain the solutions $X_\ell(\rho)$, $Y_\ell(\rho)$, $U_\ell(\rho)$, $V_\ell(\rho)$ which enter into expressions (155) and (156). Several cases for which the differential equations (103) and (104) may be solved analytically will be considered below.

B. Special Case of a Spherical Shell

We first consider the case of a spherical shell, for which the electromagnetic properties of Regions I and III are assumed to be identical ($k^{I} = k^{III}$), and Region II is assumed to be homogeneous ($n_2 = k^{II}/k^{I}$ = constant). We then have $n_3 = k^{III}/k^{I} = 1$. Eqs. (103) and (104) for W_ℓ and G_ℓ are then identical and have the previously noted solutions

$$X_\ell(\rho) = U_\ell(\rho) = \psi_\ell(n_2 \rho) \tag{157}$$

The linearly independent solutions are

$$Y_\ell(\rho) = V_\ell(\rho) = \chi_\ell(n_2 \rho) \tag{158}$$

where $\chi_\ell(z)$ is related to the Neumann function $N_{\ell+1/2}(z)$ as follows:

$$\chi_\ell(z) = \sqrt{\frac{\pi z}{2}}\, N_{\ell+1/2}(z) \tag{159}$$

The ratios defined by (154) then become

$$\mu_\ell(x) = \xi_\ell(x) = n_2 D_\ell(n_2 x) \tag{160}$$

$$\nu_\ell(x) = \eta_\ell(x) = n_2 E_\ell(x)$$

where

$$E_\ell \equiv \frac{\chi_\ell'(x)}{\chi_\ell(x)} \tag{161}$$

Substituting Eqs. (157) - (158) and (160) - (161) into expression (155) and (156), we then find

$$ {}^{e}B_{\ell} = \frac{-\psi_{\ell}(x)}{\zeta_{\ell}^{(1)}(x)} \left\{ \frac{\psi_{\ell}(n_2x)\chi_{\ell}(n_2y)\left[n_2D_{\ell}(x) - D_{\ell}(n_2x)\right]\left[n_2D_{\ell}(y) - E_{\ell}(n_2y)\right] - \psi_{\ell}(n_2y)\chi_{\ell}(n_2x)\left[n_2D_{\ell}(x) - E_{\ell}(n_2x)\right]\left[n_2D_{\ell}(y) - D_{\ell}(n_2y)\right]}{\psi_{\ell}(n_2x)\chi_{\ell}(n_2y)\left[n_2\Gamma_{\ell}(x) - D_{\ell}(n_2x)\right]\left[n_2D_{\ell}(y) - E_{\ell}(n_2y)\right] - \psi_{\ell}(n_2y)\chi_{\ell}(n_2x)\left[n_2\Gamma_{\ell}(x) - E_{\ell}(n_2x)\right]\left[n_2D_{\ell}(y) - D_{\ell}(n_2y)\right]} \right\} \tag{162} $$

$$ {}^{m}B_{\ell} = \frac{-\psi_{\ell}(x)}{\zeta_{\ell}^{(1)}(x)} \left\{ \frac{\psi_{\ell}(n_2x)\chi_{\ell}(n_2y)\left[D_{\ell}(x) - n_2D_{\ell}(n_2x)\right]\left[n_2E_{\ell}(n_2y) - D_{\ell}(y)\right] - \chi_{\ell}(n_2x)\psi_{\ell}(n_2y)\left[D_{\ell}(x) - n_2E_{\ell}(n_2x)\right]\left[n_2D_{\ell}(n_2y) - D_{\ell}(y)\right]}{\psi_{\ell}(n_2x)\chi_{\ell}(n_2y)\left[\Gamma_{\ell}(x) - n_2D_{\ell}(n_2x)\right]\left[n_2E_{\ell}(n_2y) - D_{\ell}(y)\right] - \chi_{\ell}(n_2x)\psi_{\ell}(n_2y)\left[\Gamma_{\ell}(x) - n_2E_{\ell}(n_2x)\right]\left[n_2D_{\ell}(n_2y) - D_{\ell}(y)\right]} \right\} \tag{163} $$

If desired, these equations may be written more compactly by introducing the abbreviations

$$ \delta_{\ell}(x) = n_2D_{\ell}(x) - D_{\ell}(n_2x) \tag{164} $$

$$ \epsilon_{\ell}(x) = n_2D_{\ell}(x) - E_{\ell}(n_2x) \tag{165} $$

$$ {}^{d}\gamma_{\ell}(x) = n_2\Gamma_{\ell}(x) - D_{\ell}(n_2x) \tag{166} $$

$$ {}^{e}\gamma_{\ell}(x) = n_2\Gamma_{\ell}(x) - E_{\ell}(n_2x) \tag{167} $$

$$ \delta_{\ell}^{*} = D_{\ell}(x) - n_2D_{\ell}(n_2x) \tag{168} $$

$$ \epsilon_{\ell}^{*} = D_{\ell}(x) - n_2E_{\ell}(n_2x) \tag{169} $$

$$ {}^{d}\gamma_{\ell}^{*}(x) = \Gamma_{\ell}(x) - n_2D_{\ell}(n_2x) \tag{170} $$

$$ {}^{e}\gamma_{\ell}^{*}(x) = \Gamma_{\ell}(x) - n_2E_{\ell}(n_2x) \tag{171} $$

Equations (162) and (163) then become

Equations (162) and (163) then become

$$
{}^{e}B_{\ell} = \frac{-\psi_{\ell}(x)}{\zeta_{\ell}^{(1)}(x)} \left\{ \frac{\delta_{\ell}(x)\epsilon_{\ell}(y)\psi_{\ell}(n_2x)\chi_{\ell}(n_2y) - \delta_{\ell}(y)\epsilon_{\ell}(x)\psi_{\ell}(n_2y)\chi_{\ell}(n_2x)}{{}^{d}\gamma_{\ell}(x)\epsilon_{\ell}(y)\psi_{\ell}(n_2x)\chi_{\ell}(n_2y) - {}^{e}\gamma_{\ell}(y)\delta_{\ell}(y)\psi_{\ell}(n_2y)\chi_{\ell}(n_2x)} \right\} \tag{172}
$$

$$
{}^{m}B_{\ell} = \frac{-\psi_{\ell}(x)}{\zeta_{\ell}^{(1)}(x)} \left\{ \frac{\delta_{\ell}^{*}(x)\epsilon_{\ell}^{*}(y)\psi_{\ell}(n_2x)\chi_{\ell}(n_2y) - \delta_{\ell}^{*}(y)\epsilon_{\ell}^{*}(x)\psi_{\ell}(n_2y)\chi_{\ell}(n_2x)}{{}^{d}\gamma_{\ell}^{*}(x)\epsilon_{\ell}^{*}(y)\psi_{\ell}(n_2x)\chi_{\ell}(n_2y) - {}^{e}\gamma_{\ell}^{*}(x)\delta_{\ell}^{*}(y)\psi_{\ell}(n_2y)\chi_{\ell}(n_2x)} \right\} \tag{173}
$$

Equations (172) and (173) - alternatively, Eqs. (162) and (163) - represent the scattering coefficients for the sphercal shell (case b of Fig. 1).

C. Special Case of Decreasing Refractive Index

We shall now consider the case where the refractive index in Region II decreases with r ; in particular we consider the dependence

$$
n_2 = \frac{A}{\rho} \tag{174}
$$

where A is an arbitrary complex constant. This corresponds to case (c) of Fig. 1.

However, before proceeding to solve Eqs. (103) and (104) for the specific case where n_2 is given by (174), we shall first derive some results of

more general extent concerning the analytic nature of Eqs. (103) and (104). Equation (104) is of the general form

$$\frac{d^2 G_\ell}{d\rho^2} + f(\rho) G_\ell = 0 \tag{175}$$

while Eq. (103) has an additional term involving the first derivative $dW_\ell / d\rho$. However, Eq. (103) may likewise be cast into the general form (175). Thus, if in (103) we set

$$W_\ell = \mu \overline{W}_\ell \tag{176}$$

where μ is an unknown function of ρ , we find with

$$W_\ell' = \mu \overline{W}_\ell' + \mu' \overline{W}_\ell$$

$$W_\ell'' = u \overline{W}_\ell'' + 2\mu' \overline{W}_\ell' + \mu'' \overline{W}_\ell$$

that Eq. (103) becomes (after division by μ):

$$\overline{W}_\ell'' + 2\left[\frac{\mu'}{\mu} - \frac{n'}{n}\right]\overline{W}_\ell' + \left[\frac{\mu''}{\mu} - \frac{2n'}{n}\frac{\mu'}{\mu} + n^2 - \frac{\ell(\ell+1)}{\rho^2}\right]\overline{W}_\ell = 0 \tag{177}$$

The term proportional to $\overline{W}_\ell'$ may be eliminated if we chose

$$\frac{\mu'}{\mu} = \frac{n'}{n} \qquad \text{or} \qquad \mu = Cn \tag{178}$$

where C is an arbitrary constant which may be taken equal to one without loss of generality. Equation (177) then becomes

$$\overline{W}_\ell'' + \left[\frac{n''}{n} - 2\left(\frac{n'}{n}\right)^2 + n^2 - \frac{\ell(\ell+1)}{\rho^2}\right]\overline{W}_\ell = 0 \qquad (179)$$

which is indeed of the form (175), with

$$W_\ell = n\,\overline{W}_\ell \qquad (180)$$

Moreover, for the special case where

$$\frac{n''}{n} = 2\left(\frac{n'}{n}\right)^2 \qquad (181)$$

Equation (179) for $\overline{W}_\ell$ is identical with Eq. (104) for $G_\ell(\rho)$. The functions n for which (181) is satisfied may be obtained by integrating Eq. (181). Rewriting (181) in the form

$$\frac{n''}{n'} = 2\,\frac{n'}{n} \qquad (182)$$

a first integral is obtained as

$$n' = C\,n^2 \qquad ; \qquad C \text{ arbitrary}$$

This in turn may be integrated to yield the general solution

$$n = \left(\frac{A}{\rho + D} \right) \tag{183}$$

where A and D are arbitrary constants. Thus, for any n of the form (183), we have

$$W_\ell(\rho) = n\, G_\ell(\rho) \tag{184}$$

We now return to the specific form (174) for n , which is a particular case of (183). Accordingly, we may write

$$W_\ell(\rho) = \frac{1}{\rho} G_\ell(\rho) \tag{185}$$

(since the equation for W_ℓ is linear, the constant A in Eq. (184) can be taken as unity without any loss of generality), where $G_\ell(\rho)$ must satisfy

$$\frac{d^2 G_\ell}{d\rho^2} + \frac{\left[A^2 - \ell(\ell+1)\right]}{\rho^2} G_\ell = 0 \tag{186}$$

The general solution of Eq. (186) may be found by setting

$$G_\ell = m^2 \tag{187}$$

which leads to the characteristic equation

$$m(m-1) + \left[A^2 - \ell(\ell+1)\right] = 0 \tag{188}$$

having the solutions

$$m_{1,2} = \frac{1 \pm \sqrt{1 - 4\left[A^2 - \ell(\ell+1)\right]}}{2} \tag{189}$$

Writirg

$$p = \sqrt{\left(\ell + {}^1/_2\right)^2 - A^2} \tag{190}$$

we may write

$$\left.\begin{aligned} m_1 &= \frac{1}{2} + p \\ m_2 &= \frac{1}{2} - p \end{aligned}\right\} \tag{191}$$

According to Eqs. (127) and (128), and (185), we then find

$$U_\ell(\rho) = \rho^{p+{}^1/_2} \tag{192}$$

$$V_\ell(\rho) = \rho^{-(p-{}^1/_2)} \tag{193}$$

$$X_\ell(\rho) = \rho^{p-{}^1/_2} \tag{194}$$

$$Y_\ell(\rho) = \rho^{-(p+{}^1/_2)} \tag{195}$$

The ratios defined by Eq. (154) are then easily calculated to be

$$
\left.
\begin{aligned}
\xi_\ell(\rho) &= \frac{p + 1/2}{\rho} \\
\eta_\ell(\rho) &= -\frac{p - 1/2}{\rho} \\
\mu_\ell(\rho) &= \frac{p - 1/2}{\rho} \\
\nu_\ell(\rho) &= -\frac{p + 1/2}{\rho}
\end{aligned}
\right\}
\tag{196}
$$

Substituting Eqs. (192) through (196) into the general expressions (155) through (156), we find after some simplifications that the scattering coefficients may be written in the form

$$
{}^{e}B_{\ell} = \frac{-\psi_{\ell}(x)}{\zeta_{\ell}^{(1)}(x)} \left\{ \frac{K^{2p}\left[A^2 D_{\ell}(x) - (p-\tfrac{1}{2})x\right]\left[A^2 D_{\ell}(n_3 y) + n_3(p+\tfrac{1}{2})y\right] - \left[A^2 D_{\ell}(x) + (p+\tfrac{1}{2})x\right]\left[A^2 D_{\ell}(n_3 y) - n_3(p-\tfrac{1}{2})y\right]}{K^{2p}\left[A^2 \Gamma_{\ell}(x) - (p-\tfrac{1}{2})x\right]\left[A^2 D_{\ell}(n_3 y) + n_3(p+\tfrac{1}{2})y\right] - \left[A^2 \Gamma_{\ell}(x) + (p+\tfrac{1}{2})x\right]\left[A^2 D_{\ell}(n_3 y) - n_3(p-\tfrac{1}{2})y\right]} \right\} \tag{197}
$$

$$
{}^{m}B_{\ell} = \frac{-\psi_{\ell}(x)}{\zeta_{\ell}^{(1)}(x)} \left\{ \frac{K^{2p}\left[x D_{\ell}(x) - (p+\tfrac{1}{2})\right]\left[(p-\tfrac{1}{2}) + n_3 y D_{\ell}(n_3 y)\right] + \left[x D_{\ell}(x) + (p-\tfrac{1}{2})\right]\left[(p+\tfrac{1}{2}) - n_3 y D_{\ell}(n_3 y)\right]}{K^{2p}\left[x \Gamma_{\ell}(x) - (p+\tfrac{1}{2})\right]\left[(p-\tfrac{1}{2}) + n_3 y D_{\ell}(n_3 y)\right] + \left[x \Gamma_{\ell}(x) + (p-\tfrac{1}{2})\right]\left[(p+\tfrac{1}{2}) - n_3 y D_{\ell}(n_3 y)\right]} \right\} \tag{198}
$$

where we have written $K = (b/a)$.

These expressions are relatively simple; in fact, they are much simpler than the corresponding expressions obtained in the preceding section for a spherical shell with uniform refractive index. For this reason, it was felt desirable to pay particular attention to this case, despite the fact that more general cases of decreasing refractive index (leading to exceedingly complicated expressions for ${}^{e}B_{\ell}$ and ${}^{m}B_{\ell}$) are included in the general form of $n(\rho)$ discussed in the next section.

D. Special Case of Increasing Refractive Index

1. Sphere with Central Core

Here we consider the case where the refractive index n_2 in Region II has the form

$$n_2 = A\rho^m \tag{199}$$

where A and m are arbitrary complex constants, with the exception that the case $m = -1$ is excluded (this case was treated separately in the preceding section). We shall assume that m is real and positive and that A has both positive real and imaginary parts, such that (199) describes an increasing refractive index (corresponding to curve (d) in Figure 1), although the analysis which follows applies equally well to the general case.

Equations (103) and (104) in Region II then take the form

$$\frac{d^2 W_\ell}{d\rho^2} - \frac{2m}{\rho}\,\frac{dW_\ell}{d\rho} + \left[A^2 \rho^{2m} - \frac{\ell(\ell+1)}{\rho^2}\right] W_\ell = 0 \tag{200}$$

$$\frac{d^2 G_\ell}{d\rho^2} + \left[A^2 \rho^{2m} - \frac{\ell(\ell+1)}{\rho^2}\right] W_\ell = 0 \tag{201}$$

Both of these equations are particular forms of the more general equation

$$x^2 y'' + axy' + (bx^r + c)y = 0 \tag{202}$$

whose solution (for $r \neq 0$, $b \neq 0$) is given by Kamke[5] as

$$y = x^{\frac{(1-a)}{2}} Z_\nu\left(\frac{2}{r} \sqrt{b}\, x^{\frac{r}{2}}\right) \tag{203}$$

where

$$\nu = \frac{1}{r}\sqrt{(1-a)^2 - 4c} \neq 0 \tag{204}$$

and where Z_ν is any one of the Bessel functions.

Thus, Eq. (200) is solved by making the identifications

$$x = \rho , \quad y = W_\ell , \quad a = -2m , \quad b = A^2 , \quad r = 2m , \quad c = -\ell(\ell+1) \tag{205}$$

Substituting (205) into (203) and (204) we accordingly find that the two independent solutions of W_ℓ are

$$X_\ell(\rho) = \rho^m \sqrt{\frac{\pi\rho}{2}}\, J_\nu\left(\frac{A}{m+1}\, \rho^{m+1}\right) \tag{206}$$

$$Y_\ell(\rho) = \rho^m \sqrt{\frac{\pi\rho}{2}}\, J_{-\nu}\left(\frac{A}{m+1}\, \rho^{m+1}\right) \tag{207}$$

where

$$\nu = \frac{1}{m+1} \sqrt{\ell(\ell+1) + (m+\tfrac{1}{2})^2} \tag{208}$$

Similarly, if we make the identifications

$$x = \rho \ , \quad y = G_\ell \ , \quad a = 0 \ , \quad b = A^2 \ , \quad r = 2m \ , \quad c = -\ell(\ell+1) \tag{209}$$

Eq. (202) becomes Eq. (104) for G_ℓ, and the two independent solutions of G_ℓ are found to be

$$U_\ell(\rho) = \sqrt{\frac{\pi\rho}{2}}\, J_\mu\left(\frac{A}{m+1}\, \rho^{m+1}\right) \tag{210}$$

$$V_\ell(\rho) = \sqrt{\frac{\pi\rho}{2}}\, J_{-\mu}\left(\frac{A}{m+1}\, \rho^{m+1}\right) \tag{211}$$

where

$$\mu = \frac{2\ell+1}{2(m+1)} \tag{212}$$

The scattering coefficients are obtained by substituting expressions (206) through (211) into Eqs. (155) and (156), as well as into the expressions (154) which enter into the latter. Inasmuch as the functions X_ℓ, Y_ℓ, U_ℓ, V_ℓ as given by Eqs. (206) through (211) are exceedingly complicated, no significant simplifications are possible. We note also that for the first time we are faced with Bessel functions both whose argument and

order may be complex. Moreover, inasmuch as the index ℓ enters in a complicated manner into the orders μ and ν of the Bessel functions, it is impossible to write recursion formulas in ℓ for the Bessel functions involved.

Finally, we note that the solution presented in this section has already been previously obtained by Levine and Kerker[2] (cf. also Namura and Takaku[6]), although their expressions for the potentials and scattering coefficients contain some errors.

2. Sphere without a Central Core

Finally we consider the case where the refractive index is given by (199) throughout the entire sphere, i.e. where the central core of constant refractive index is absent. This corresponds to the dotted curve in case (d) of Fig. 1.

Only those solutions having no singularities at the origin are then admissible, and we have

$$\begin{aligned} W_\ell(\rho) &= X_\ell(\rho) \\ G_\ell(\rho) &= U_\ell(\rho) \end{aligned} \tag{213}$$

where $X_\ell(\rho)$ and $U_\ell(\rho)$ are given by (206) and (210). The scattering coefficients are then obtained by substituting (213) and (199) into the general expressions (117) and (118), and we obtain

$$ {}^{e}B_\ell = \frac{\psi_\ell(x)}{\zeta_\ell^{(1)}(x)} \left[\frac{A^2 D_\ell(x) - x^{-2m}\omega_\ell(x)}{x^{-2m}\omega_\ell(x) - A^2\Gamma_\ell(x)} \right] \tag{214} $$

$$ {}^{m}B_\ell = \frac{\psi_\ell(x)}{\zeta_\ell^{(1)}(x)} \left[\frac{\gamma_\ell(x) - D_\ell(x)}{\Gamma_\ell(x) - \gamma_\ell(x)} \right] \tag{215} $$

where

$$ \omega_\ell(x) = \frac{X_\ell'(x)}{X_\ell(x)} $$
$$ \gamma_\ell(x) = \frac{U_\ell'(x)}{U_\ell(x)} \tag{216} $$

As in the preceding section, these expressions are intrinsically quite complicated and no further reduction is possible.

VII. DERIVATION OF THE RADAR CROSS-SECTION

In the preceding sections, we have obtained exact analytical expressions for the scattering coefficients ${}^{e}B_{\ell}$ and ${}^{m}B_{\ell}$ for a variety of cases. Our ultimate interest is to determine the radar cross-sections of the various objects under study, and it is the aim of the present section to show that the radar cross-section can in general be expressed entirely in terms of the scattering coefficients ${}^{e}B_{\ell}$ and ${}^{m}B_{\ell}$.

The radar cross-section is defined as $4\pi r^2$ times the ratio of the Poynting vector of the wave scattered in the negative z-direction (i.e., the radial component of the Poynting vector for $\theta=\pi$) to the Poynting vector of the incident wave traveling in the positive z-direction. Accordingly, we may write the radar cross-section as

$$\sigma_R = - \frac{4\pi r^2 \, \Pi_r^s \big|_{\theta=\pi}}{\Pi_z^i} \tag{217}$$

where the minus sign is introduced in order to make the cross-section a positive quantity. The Poynting vectors entering into Eq. (217) are

$$\Pi_r^{\,s}\Big|_{\theta=\pi} = \frac{1}{2}\,\mathrm{Re}\left(\vec{E}^{\,s} \times \vec{H}^{\,s*}\right)_r\Big|_{\theta=\pi} \tag{218}$$

$$\Pi_z^{\,i} = \frac{1}{2}\,\mathrm{Re}\left(\vec{E}^{\,i} \times \vec{H}^{\,i*}\right)_z \tag{219}$$

where the superscripts i, s, * stand for incident, scattered and complex conjugate, respectively, and where the subscripts indicate the appropriate vector component.

We first turn our attention to the Poynting vector of the scattered wave, which may be written in the expanded form

$$\Pi_r^{\,s} = \frac{1}{2}\,\mathrm{Re}\left(E_\theta^{\,s} H_\varphi^{\,s*} - E_\varphi^{\,s} H_\theta^{\,s*}\right) \tag{220}$$

The field components required in Eq. (220) are in general found from expressions (54), (55), (57) and (58) with ${}^{e}\Omega$ and ${}^{m}\Omega$ given by (86) and (87). Thus, for example, if we consider $E_\varphi^{\,s}$, we obtain

$$E_\varphi^{\,s} = -\frac{\sin\varphi}{kr}\sum_{\ell=1}^{\infty}\frac{i^{\ell-1}(2\ell+1)}{\ell(\ell+1)}\left\{{}^{e}B_\ell\,\zeta_\ell^{(1)\prime}(kr)\,\frac{P_\ell^1(\cos\theta)}{\sin\theta} + {}^{m}B_\ell\, i\,\zeta_\ell^{(1)}(kr)\,\frac{dP_\ell^1(\cos\theta)}{d\theta}\right\} \tag{221}$$

where we have written $k^I = k$ and have made use of the relation $k_1^I k_2 = -k^2$. Similarly, we obtain

$$E_\theta^s = \frac{\cos\varphi}{kr} \sum_{\ell=1}^{\infty} \frac{i^{\ell-1}(2\ell+1)}{\ell(\ell+1)} \left\{ {}^eB_\ell\, \zeta_\ell^{(1)\prime}(kr) \frac{dP_\ell^1(\cos\theta)}{d\theta} + i\, {}^mB_\ell\, \zeta_\ell^{(1)}(kr) \frac{P_\ell^1(\cos\theta)}{\sin\theta} \right\} \tag{222}$$

$$H_\varphi^s = \frac{-\cos\varphi}{k_2 r} \sum_{\ell=1}^{\infty} \frac{i^{\ell-1}(2\ell+1)}{\ell(\ell+1)} \left\{ {}^eB_\ell\, \zeta_\ell^{(1)}(kr) \frac{dP_\ell^1(\cos\theta)}{d\theta} - i\, {}^mB_\ell\, \zeta_\ell^{(1)\prime}(kr) \frac{P_\ell^1(\cos\theta)}{\sin\theta} \right\} \tag{223}$$

$$H_\theta^s = \frac{-\sin\varphi}{k_2 r} \sum_{\ell=1}^{\infty} \frac{i^{\ell-1}(2\ell+1)}{\ell(\ell+1)} \left\{ {}^eB_\ell\, \zeta_\ell^{(1)}(kr) \frac{P_\ell^1(\cos\theta)}{\sin\theta} - i\, {}^mB_\ell\, \zeta_\ell^{(1)\prime}(kr) \frac{dP_\ell^1(\cos\theta)}{d\theta} \right\} \tag{224}$$

We are generally interested in the radar cross-section for the case where the receiver is located at large distances from the scattering object (far-field); accordingly, we may replace the Ricatti-Hankel functions by their asymptotic forms:

$$\zeta_\ell^{(1)}(kr) = (-i)^{\ell+1}\, e^{ikr} \tag{225}$$

$$\zeta_\ell^{(1)\prime}(kr) = (-i)^{\ell}\, e^{ikr} = i\zeta_\ell^{(1)}(kr) \tag{226}$$

Furthermore, we are interested in the field components for the particular value $\theta = \pi$. Thus, by making use of the well-known relations

$$\left.\frac{dP_\ell^1(\cos\theta)}{d\theta}\right|_{\theta=\pi} = -\left.\frac{P_\ell^1(\cos\theta)}{\sin\theta}\right|_{\theta=\pi} = (-1)^\ell \frac{\ell(\ell+1)}{2} \tag{227}$$

expressions (221) through (224) for the required field components become

$$E_\varphi^s = -\frac{ie^{ikr}\sin\varphi}{kr}\sum_{\ell=1}^{\infty}(-1)^\ell\left(\ell+\frac{1}{2}\right)\left({}^eB_\ell - {}^mB_\ell\right) \tag{228}$$

$$E_\theta^s = -\frac{ie^{ikr}\cos\varphi}{kr}\sum_{\ell=1}^{\infty}(-1)^\ell\left(\ell+\frac{1}{2}\right)\left({}^eB_\ell - {}^mB_\ell\right) \tag{229}$$

$$H_\varphi^s = \frac{e^{ikr}\cos\varphi}{k_2 r}\sum_{\ell=1}^{\infty}(-1)^\ell\left(\ell+\frac{1}{2}\right)\left({}^eB_\ell - {}^mB_\ell\right) \tag{230}$$

$$H_\theta^s = -\frac{e^{ikr}\sin\varphi}{k_2 r}\sum_{\ell=1}^{\infty}(-1)^\ell\left(\ell+\frac{1}{2}\right)\left({}^eB_\ell - {}^mB_\ell\right) \tag{231}$$

Substituting these expressions into Eq. (220) for the Poynting vector, and making use of the usual rule for expressing a product of two infinite

series as a doubly-infinite series, we obtain

$$\Pi_r^s\Big|_{\theta=\pi} = \frac{1}{2}\,\mathrm{Re}\left\{\frac{-i}{k k_2 r^2}\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{\ell}(-1)^{m}\left(\ell+\frac{1}{2}\right)\left(m+\frac{1}{2}\right)\left({}^{e}B_{\ell}-{}^{m}B_{\ell}\right)\left({}^{e}B_{\ell}^{*}-{}^{m}B_{\ell}^{*}\right)\right\}$$

$$= -\frac{1}{2k\,|k_2|\,r^2}\left|\sum_{\ell=1}^{\infty}(-1)^{\ell}\left(\ell+\frac{1}{2}\right)\left({}^{e}B_{\ell}-{}^{m}B_{\ell}\right)\right|^2 \tag{232}$$

where use has been made of the fact that $k_2 = i\,|k_2|$ and where the bars denote absolute value.

The incident Poynting vector Π_z^i is easily found from Eqs. (219) with the incident fields given by (64), and we obtain

$$\Pi_z^i = \frac{1}{2}\,\frac{k}{|k_2|} \tag{233}$$

Substituting expressions (232) and (233) into (217), we finally obtain the desired expression for the radar cross-section:

$$\sigma_R = \frac{4\pi}{k^2}\left|\sum_{\ell=1}^{\infty}(-1)^{\ell}\left(\ell+\frac{1}{2}\right)\left({}^{e}B_{\ell}-{}^{m}B_{\ell}\right)\right|^2 \tag{234}$$

In conclusion, we also cite the corresponding results for the extinction, total scattering, and absorption cross-sections (cf., for example Wyatt[1]):

$$\sigma_{ext} = \frac{4\pi}{k^2} \mathrm{Re} \sum_{\ell=1}^{\infty} \left(\ell + \frac{1}{2}\right)\left({}^{e}B_{\ell} + {}^{m}B_{\ell}\right) \tag{235}$$

$$\sigma_{scat} = \frac{4\pi}{k^2} \sum_{\ell=1}^{\infty} \left(\ell + \frac{1}{2}\right)\left(\left|{}^{e}B_{\ell}\right|^2 + \left|{}^{m}B_{\ell}\right|^2\right) \tag{236}$$

$$\sigma_{abs} = \sigma_{ext} - \sigma_{scat} \tag{237}$$

VIII. CONCLUSION

As described more fully in the Introduction, the purpose of the present investigation is to determine whether the measurement of radar cross-section profiles is a potentially useful diagnostic tool for ascertaining the electron density distribution of inhomogeneous plasma spheres of practical interest. Toward this end, we have obtained analytical expressions for the radar cross-sections (at arbitrary frequency) of some typical examples of spherically symmetric plasma spheres with increasing and decreasing refractive index, as a function of radial distance from the origin. Specifically, we have considered the four different electron density distributions illustrated schematically in Figure 1.

We have found that in each case, the radar cross-section is completely defined by means of two sets of scattering coefficients ${}^{e}B_{\ell}$ and ${}^{m}B_{\ell}$, in terms of which the radar cross-section can be calculated by Eq. (234) of Section VII. Accordingly, we have obtained analytical expressions for the coefficients ${}^{e}B_{\ell}$ and ${}^{m}B_{\ell}$ for each of the four schematic cases

illustrated in Fig. 1. These expressions which constitute the chief results of the present report, are given by Eqs. (123) and (124) for case (a), by Eqs. (172) and (173) for case (b), by Eqs. (197) and (198) for case (c), and finally by Eqs. (155), (156) (together with (206)-(211)) and (214), (215) for the discontinuous and continuous distributions of case (d), respectively.

We wish to emphasize that the expressions obtained for the scattering coefficients are exact and are based on a full wave treatment of the scattering problem, without recourse to any mathematical approximations. Moreover, our final results are expressed entirely in terms of well-known analytical functions. However, these expressions - while of varying complexity - are in general too cumbersome for hand calculation. Thus, it is likely that a considerable number of terms are required in the infinite series (234) defining the radar cross-section in order to achieve the desired numerical accuracy. Moreover, the problem of numerical evaluation is complicated by the fact that not all of the required Bessel functions are readily available in tabulated form. For these reasons, numerical evaluation of our results will require the use of a computer.

This numerical reduction of our results, as well as the derivation of asymptotic analytical expressions for the limiting cases of very high

and very low frequencies, will constitute the subject matter of Part II of the present investigation.

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